Gradually Varied Flow in a Trapezoidal Channel, towards a Hydraulics with Fractional Calculation and Finite Differences

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Abstract: *Several engineering problems involve use a derivative of a function; in recent years, physical phenomena have been reviewed are traditionally considered to be derived with an integer order in their mathematical models, instead derived with fractional order. Gradually varied flow evaluation profile in a free-surface channel is traditionally resolved with differential calculus and integer derivatives solved with finite difference schemes. In this work, equations were obtained relate derivatives of half order derivative of first order for different algebraic functions, resulting in polynomial type models; since there is a proportion between first derivative with derivatives of fractional order through these expressions and observing flow profiles can adjust a y(x) function approximately polynomial it is established various tests must be done to approximate to gradually varied flow equation solution help fractional differential equations can be solved through fractional finite differences.*

Keywords: *Fractional calculation, gradually varied flow, trapezoidal channel, hydraulics, half derivative*

1. Introduction

Gradually varied flow profiles determination has traditionally been approached with first derivative of strut with respect to length *x* recorded by flow [1] (formula 1).

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \tag{1}$$

Where

S_0, channel bottom slope, dimensionless
S_f, hydraulic slope, dimensionless
Fr, Froude number
$\frac{dy}{dx}$ Deep y first derivative with respect to chain x.

Numerical methods of the Euler method type, Runge-Kutta [2], in finite differences used to solve differential equation are traditionally used with proven results with laboratory tests; direct step method is a scheme that also provides solution to this problem. Currently, fractional calculation has begun to be studied to find solutions to these problems under approach to dependent variable behavior with respect to dependent implies a variation of non-integer order.

In recent years, questions have been raised regarding derivatives consideration of integer order in mechanical and engineering problems such as free fall, simple pendulum movement, parabolic launch; soil drag phenomenon problem in diffusion wave problems, before these analysis studies and hydraulics area, diffusion equation studies resolution were made using fractional calculation [3,4,5,6,7,8].
An interpretation fractional order derivatives geometric is not yet clear [9], graphs have been made to represent them, but an interpretation curves obtained (Figure 1) has not been fully achieved.

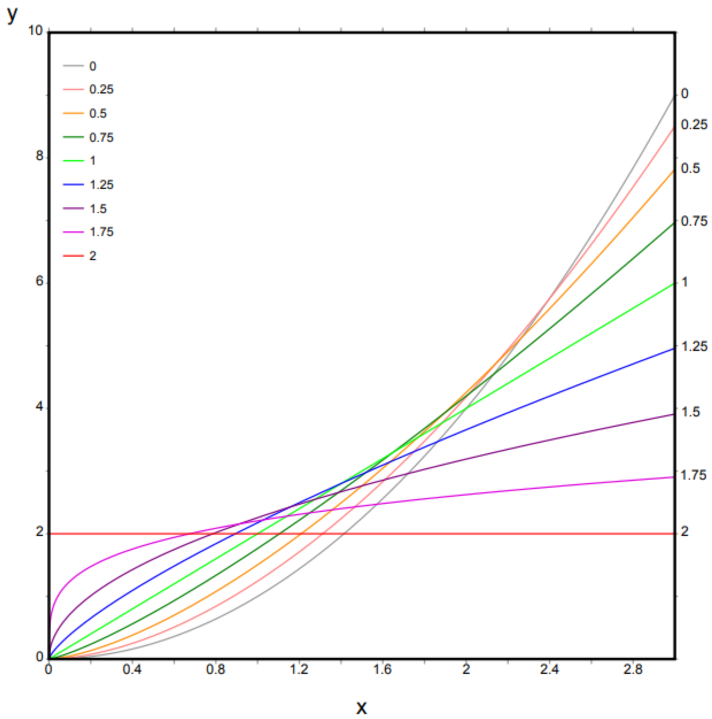

Fig. 1. Derivative of different fractional Behavior order in polynomial equation $y = x^2$. Source page 84 [3] and page 63 [9]

In this paper is proposed to obtain various functions that relate the behavior of a half derivative respect to a first order derivative of simple polynomial functions of degree p; when these functions were found, it was possible to obtain how a derivative of fractional order relates with a derivative of first order which allows this result to be transferred to a differential equation of first order to transform it into a differential equation of fractional order that can be solved with a scheme in finite fractional differences [4]. Rectangular laboratory channel geometry data were taken into account, in which an expense was passed and measurements of its braces were taken with a certain ΔX chain; adjustment was made profile measured best trend line by observing a polynomial behavior.

The method of Runge Kutta [2] was used with data for a hypothetical trapezoidal canal example [1]. This flow profile was obtained by solving a gradually varied flow differential equation; a polynomial type trend line was adjusted to profile obtained. Taking into account profiles can have polynomial behaviors and on the other hand a derivative of order one keeps a direct proportion with fractional derivative, it is possible to propose use of fractional derivatives to find an approximate solution to gradually varied flow profiles.

2. Methodology

2.1 Relationships between fractional derivative of order p and derivative of order one in polynomial functions

Santamaría, 2019 [10] when carrying out the GeoGebra © program [11] and half derivative to polynomial function are obtaining from $y=x^p$ form found two important observations:

a) As the order of the polynomial increases, the first order derivative tends to be equal to original function, as does the half order derivative
b) While first derivative has exponent p-1, half derivative has exponent p-0.5, it means first order derivative is equal to the half derivative raised to (p-1) / (p-0.5) and then divided by a constant determined with the gamma function as follows (formula 2):

$$D_x y = C\left(D_x^{1/2} y\right)^{\frac{p-1}{p-0.5}} \quad (2)$$

Half derivative is no longer valid around p = 170.25, but it remains to be verified if this is due to the used expression involving gamma function or is a software limitation.

On the other hand, when it drawing half behavior derivative respect to first order derivative polynomial functions degrees two and three, polynomial trend lines were determined (Figures 2 and 3).

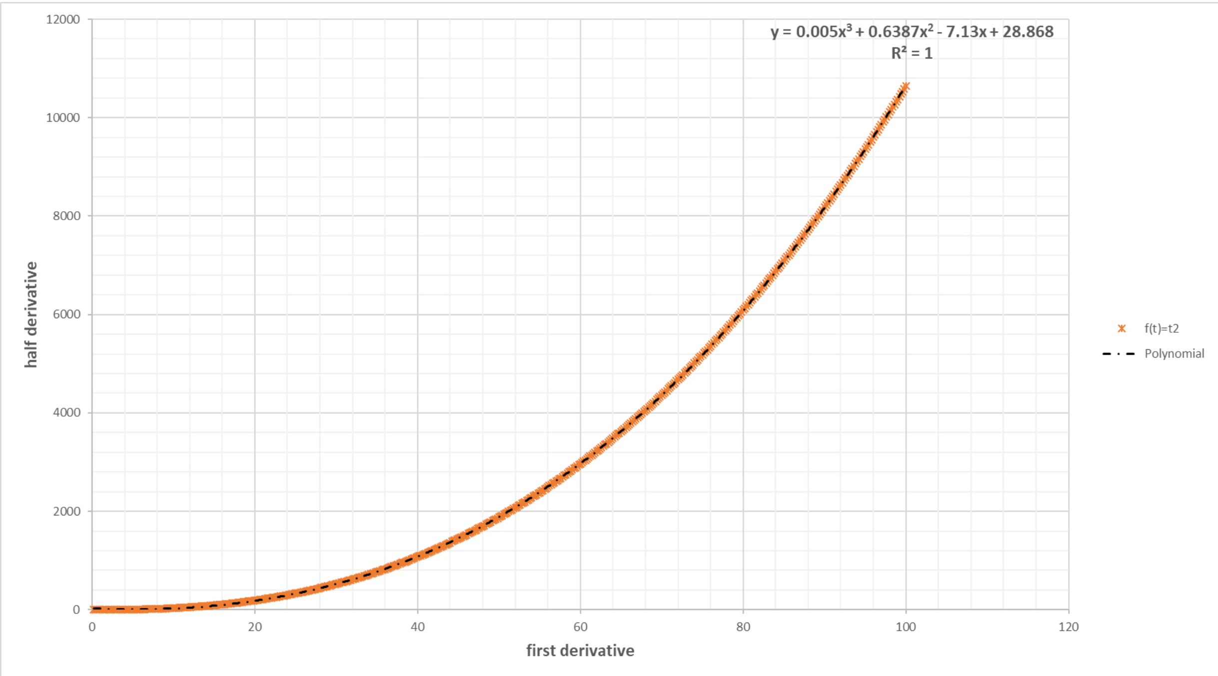

Fig. 2. Half derivative Graph vs first order derivative for function $y = x^2$

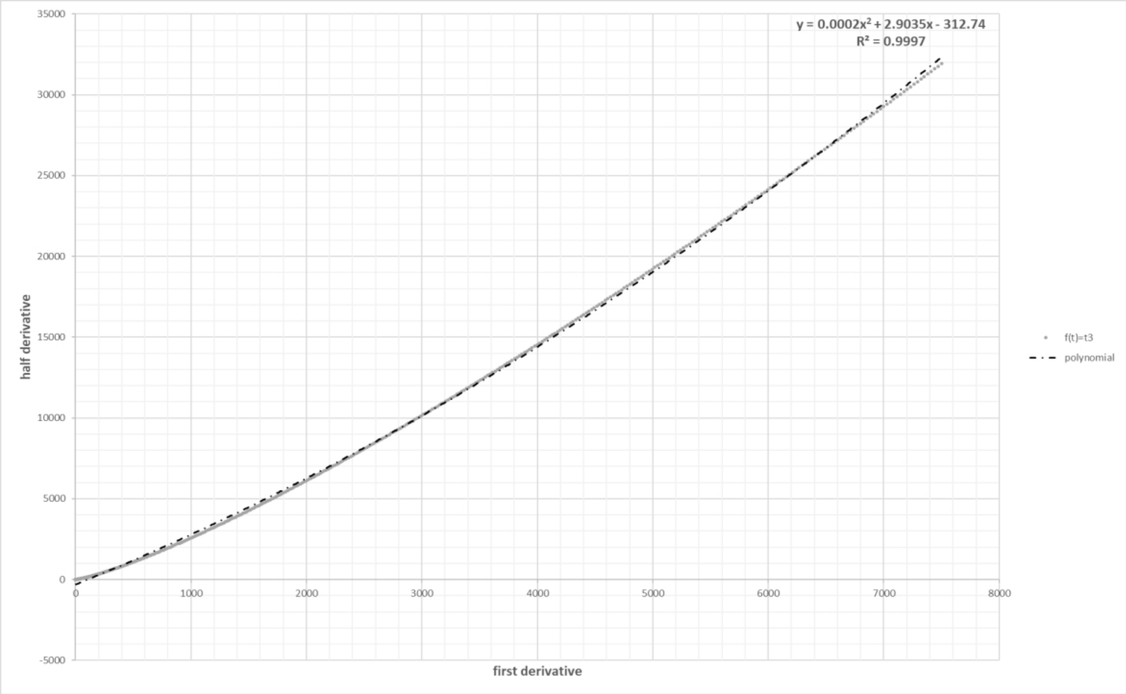

Fig. 3. Half derivative Graph vs first order derivative for function $y = x^3$

2.2 Theoretical gradually varied flow profile behavior

When taking into account measured data in laboratory from trapezoidal canal gradually varied flow profile and looking for trend line between variable x and strap and it was found the best curved portion profile trend line is a third-grade polynomial (Figure 4).

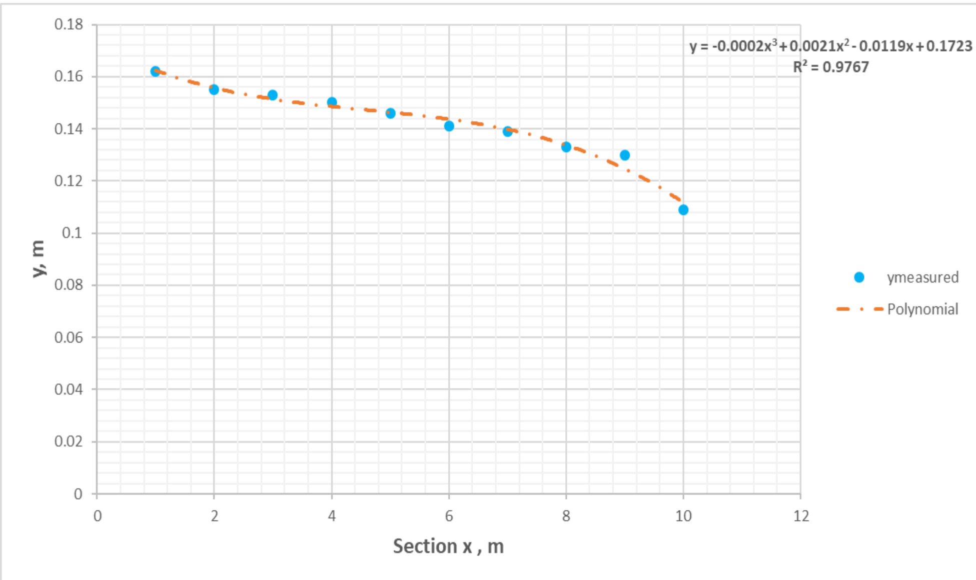

Fig. 4. Y Deep behavior Graph vs x chain in a laboratory trapezoidal channel

When considering data from hypothetical trapezoidal channel and numerical estimation performing to tie and against x-chain using Runge Kutta numerical method, a trend line corresponding to a degree two polynomial with a determination high coefficient is obtained (Figure 5).

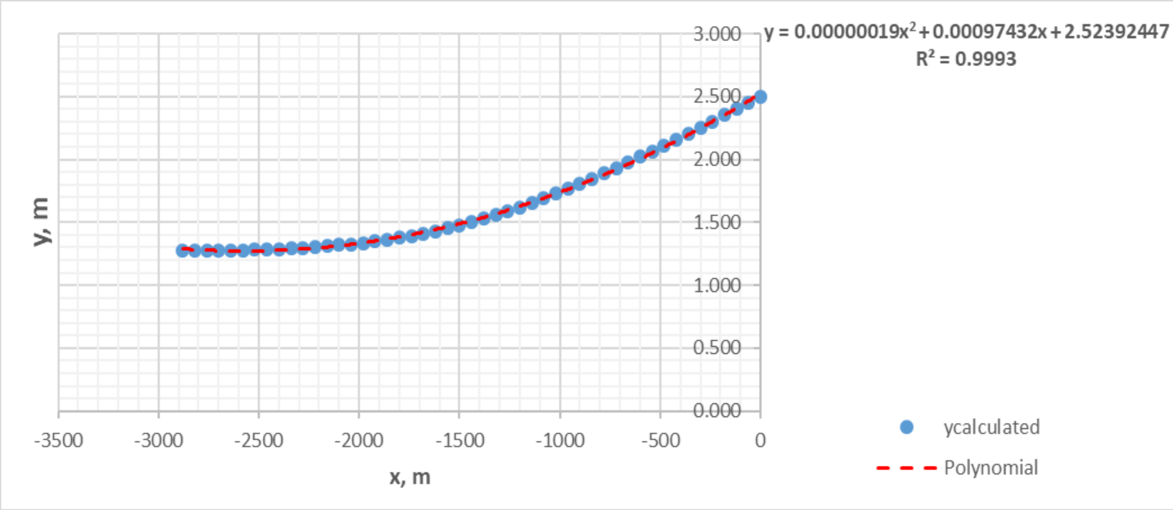

Fig. 5. Y Deep behavior Graph vs x chain in a hypothetical trapezoidal channel

3. Conclusions

Analyzing figures 4 and 5 we can conclude that flow profile can approximated by a polynomial function of grades between 1 and 3, so the first order derivative function y'(x) can be related to a fractional derivative of half order $y^{1/2}(x)$ according to equation 1 or with transformations expressed by equations 2 and 3. Schemes in fractional finite differences such as cited in Appendix A can also be used to approximate gradually varied flow profile solution.

4. Appendix A

A.1 Fourth order Runge Kutta method

Runge Kutta's methods numerically solve a first-order ordinary differential equation with initial conditions to form [2] (formula A.1):

$$y' = f(x,y); \text{ Subject to } y(x_0) = y_0 \tag{A.1}$$

In a range of values $x_0 \leq x \leq x_n$, it considering a constant increment h in independent variable x.

To do this, use following recurrence equations applied at each solution point:

$$\begin{gathered} k_1 = f(x_i, y_i) \\ k_2 = f(x_i + 0.5h, y_i + k_1 h) \\ k_3 = f(x_i + 0.5h, y_i + k_2 h) \\ k_4 = f(x_i + h, y_i + k_3 h) \\ y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 3k_3 + k_4) \\ i = 0,1,2,..n-1 \end{gathered} \tag{A.2}$$

A2. Finite fractional differences corrective predictive method

Corrective predictive method obtains differential equation numerical solution of fractional order [11]:

$$cD_o^{\alpha}u(t) = f(t, u(t)), t \in [0, T] \tag{A.3}$$

Subject to initial condition

$$u^{(j)}(o) = u_0^{(j)}, j = 0,1,2, \dots, n-1$$

Where α> 0, n is first integer not less than α

Form of a corrective predictive method, obtained from Adams-Bashford-Moulton method is:

$$u_{k+1}^{p} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} u_0^{j} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, u_j) \tag{A.4}$$

$$u_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} u_0^{j} + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{k} a_{j,k+1} f(t_j, u_j) + a_{k+1,k+1} f(t_{k+1}, u_{k+1}^{p})\right] \tag{A.5}$$

Where

$$a_{j,k+1} = \frac{\Delta t^{\alpha}}{\alpha(\alpha+1)} \begin{Bmatrix} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}, \quad j = 0 \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, 1 \le j \le k \\ 1, \quad j = k+1 \end{Bmatrix} \tag{A.6}$$

and

$$b_{j,k+1} = \frac{\Delta t^{\alpha}}{\alpha}[(k-j+1)^{\alpha} - (k-j)^{\alpha}], \qquad 0 \le j \le k \tag{A.7}$$

Where
u_j Is approximate solution of $u(t_j)$

This method applies prediction equation (A.4) to estimate an approximate value of u_{k+1} subsequently used in corrective equation (A.5) to obtain an improved approximation value.

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