ISSN 1453 – 7303 "HIDRAULICA" (No. 4/2020)
Magazine *of Hydraulics, Pneumatics, Tribology, Ecology, Sensorics, Mechatronics*

Mathematics for Real-Time S-Curve Profile Generator

PhD Eng. **Marian BLEJAN[1*]**, student **Robert BLEJAN[2]**

[1] Hydraulics and Pneumatics Research Institute INOE 2000-IHP

[2] University POLITEHNICA of Bucharest, Faculty of Electrical Engineering

* blejan.ihp@fluidas.ro, robertblejan@yahoo.com

Abstract: *Position control is usually achieved using a position controller and a profile generator. The motion profile generator establishes the trajectory based on reference position and imposed motion constrains, and the position controller forces the actual position to trace the generated position trajectory. The S-curves motion profile generator allows control for the maximum values of speed, acceleration and jerk. When profile position value reaches the value of final point position, the generator sends a signal to the set-point generator to move on to next desired point of motion profile, and so on. This motion profile generator will be implemented as a software sequence which is executed in real-time. The software of motion profile generator will be validated by comparison with the simulation in MATLAB for S-curve motion.*

Keywords: *S-curve motion, profile generator, real-time software, mathematics*

1. Introduction

In many mechatronic and robotic applications, it is necessary to perform the movement from one point to another, a point-to-point movement profile of the third order is used [1]. This paper provides analytical formulas to calculate the fastest movement between two points, based on the given maximum levels of speed, acceleration, jerk and jump value.
In the chapter two we study the three-order movement profiles possible for the movement between two points. Based on the general equations of motion for position, velocity, acceleration and shock, the six phases of motion identified by the time values *0, t1 .. t6, t7* were determined. The general equations are customized for each phase of the motion, resulting in six sets of equations that describe the motion [2].
The next chapter is intended to choose the optimal movement profile and determine the durations of the movement phases. The profile selection criterion is the minimum duration of the movement in the conditions of performing the desired jump with the limitations of speed, acceleration and shock. The parameters v_a, s_a si s_v are defined and calculated, which in correlation with the desired jump value and with the values of the speed, acceleration and shock limits allow the choice of the optimal movement profile. Thus, six possible movement profiles are identified. The equations of motion and the durations of the phases of motion are determined for each possible motion profile.
Chapter four is dedicated to the description of the original algorithm for generating the motion profile in real time. The motion algorithm is a software sequence that is called by the main program every time the input of the motion profile generator changes, for example it can be called at each reference sampling. Thus, the algorithm is developed according to the real-time programming paradigm "Time-driven programming" [3], at each change of the input the generator calculates the motion profile at the current time which is the set of values for position, speed, acceleration and shock.
Chapter five evaluates the S-curve motion profile generation algorithm by comparing the motion profiles generated by the proposed algorithm and a completely built simulation model in MATLAB. The chosen assessment criterion was the value of the position error for the final position of the generated and simulated motion profiles. A full assessment was performed for all six cases of possible movement profiles.
Finally, the results obtained are analysed and the directions to be followed for further research are proposed.

2. Definition of terms and mathematical description of the movement profile

The general equations of motion are:

$$j(t); can\ be\ J, -J\ or\ 0 \tag{1}$$

$$a(t) = a_0 + j(t) * t \tag{2}$$

$$v(t) = v_0 + a_0 * t + j(t) * \frac{t^2}{2} \tag{3}$$

$$p(t) = p_0 + v_0 * t + a_0 * \frac{t^2}{2} + j(t) * \frac{t^3}{6} \tag{4}$$

Note: it is enough to calculate the trajectory up to half the movement time $t_{miscare}$ and the position is $\frac{s}{2}$, the rest of the trajectory is symmetrical horizontally, with respect to the y-axis $y(t) = \frac{t_{miscare}}{2}$ and vertically with respect to the y-axis $y(t) = \frac{s}{2}$. The jump, the amplitude of the movement, is denoted by *s*.

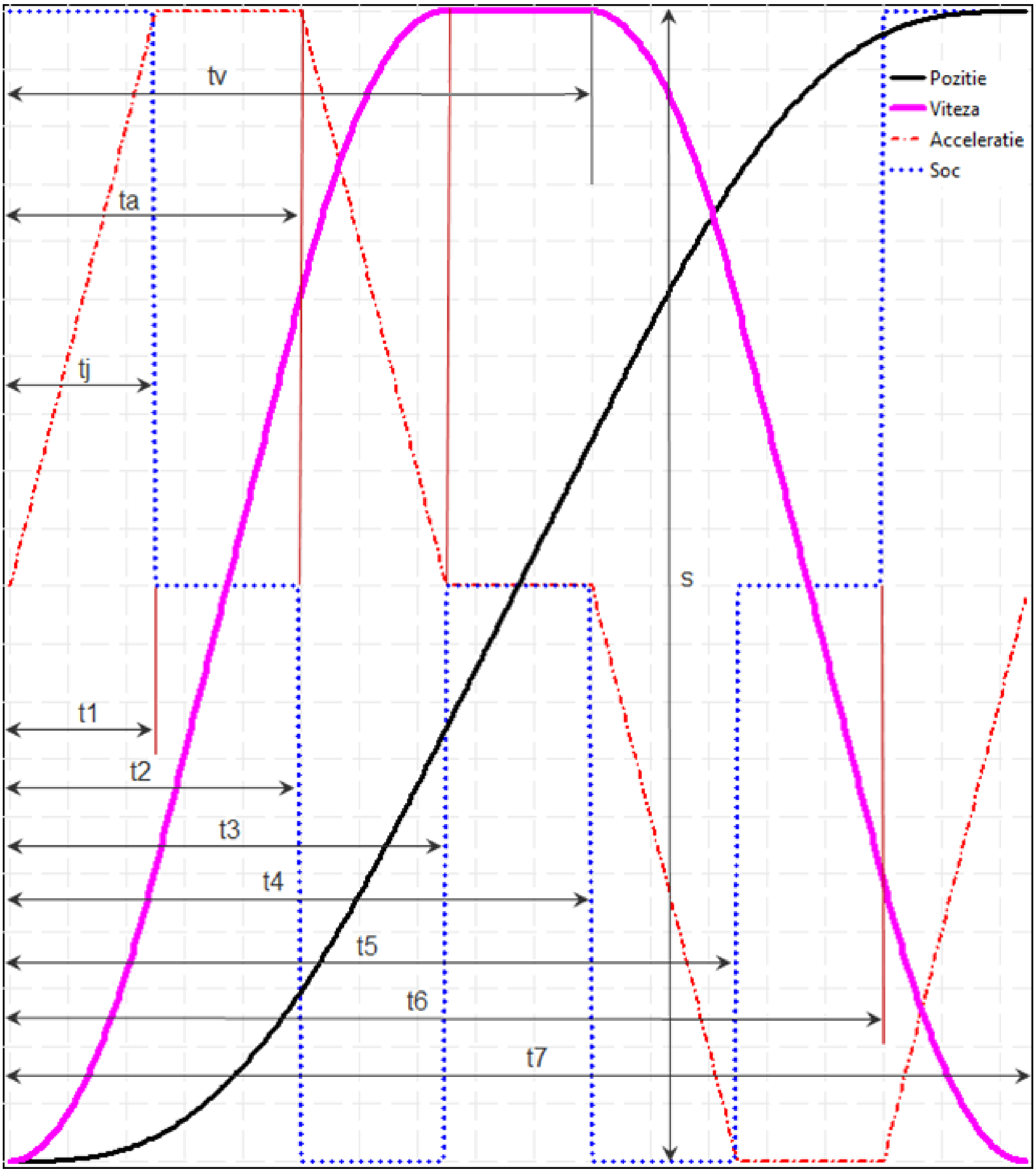

Fig. 1. Defining times t_j, t_a, t_v, $t_{1..7}$ and the amplitude of the jump *s*

Equations of motion as a function of time, see fig. 1, are:

$[0, t_1) \Rightarrow$

$$j(t) = J \tag{5}$$

$$a(t) = J * t;\ a_1 = J * t_1 \tag{6}$$

$$v(t) = J * \frac{t^2}{2};\ \ v_1 = J * \frac{{t_1}^2}{2} \tag{7}$$

$$p(t) = J * \frac{t^3}{6};\ \ p_1 = J * \frac{{t_1}^3}{6} \tag{8}$$

$[t_1, t_2) \Rightarrow$

$$j(t - t_1) = 0 \tag{9}$$

$$a(t - t_1) = a_1;\ a_2 = a_1 \tag{10}$$

$$v(t - t_1) = v_1 + a_1 * (t - t_1);\ \ v_2 = v_1 + a_1 * (t_2 - t_1) \tag{11}$$

$$p(t - t_1) = p_1 + v_1 * (t - t_1) + a_1 * \frac{(t - t_1)^2}{2};\ \ p_2 = p_1 + v_1 * (t_2 - t_1) + a_1 * \frac{(t_2 - t_1)^2}{2} \tag{12}$$

$[t_2, t_3) \Rightarrow$

$$j(t - t_2) = -J \tag{13}$$

$$a(t - t_2) = a_2 - J * (t - t_2);\ a_3 = a_2 - J * (t_3 - t_2) = 0 \tag{14}$$

$$v(t - t_2) = v_2 + a_2 * (t - t_2) - J * \frac{(t - t_2)^2}{2};\ \ v_3 = v_2 + a_2 * (t_3 - t_2) - J * \frac{(t_3 - t_2)^2}{2} \tag{15}$$

$$p(t - t_2) = p_2 + v_2 * (t - t_2) + a_2 * \frac{(t - t_2)^2}{2} - J * \frac{(t - t_2)^3}{6};$$

$$p_3 = p_2 + v_2 * (t_3 - t_2) + a_2 * \frac{(t_3 - t_2)^2}{2} - J * \frac{(t_3 - t_2)^3}{6} \tag{16}$$

$[t_3, t_4) \Rightarrow$

$$j(t - t_3) = 0 \tag{17}$$

$$a(t - t_3) = 0;\ a_4 = 0 \tag{18}$$

$$v(t - t_3) = v_3;\ \ v_4 = v_3 \tag{19}$$

$$p(t - t_3) = p_3 + v_3 * (t - t_3);\ \ p_4 = p_3 + v_3 * (t_4 - t_3) \tag{20}$$

$[t_4, t_5) \Rightarrow$

$$j(t - t_4) = -J \tag{21}$$

$$a(t - t_4) = -J * (t - t_4);\ a_5 = -J * (t_5 - t_4) = -a_1 \tag{22}$$

$$v(t - t_4) = v_4 - J * \frac{(t - t_4)^2}{2};\ \ v_5 = v_4 - J * \frac{(t_5 - t_4)^2}{2} \tag{23}$$

$$p(t - t_4) = p_4 + v_4 * (t - t_4) - J * \frac{(t - t_4)^3}{6};$$

$$p_5 = p_4 + v_4 * (t_5 - t_4) - J * \frac{(t_5 - t_4)^3}{6} \tag{24}$$

$[t_5, t_6) \Rightarrow$

$$j(t - t_5) = 0 \tag{25}$$

$$a(t - t_5) = a_5;\ a_6 = a_5 = -a_1 \tag{26}$$

$$v(t - t_5) = v_5 - a_{max} * (t - t_5);\ \ v_6 = v_5 - a_{max} * (t_6 - t_5) \tag{27}$$

$$p(t - t_5) = p_5 + v_5 * (t - t_5) + a_5 \frac{(t - t_5)^2}{2};\ \ p_6 = p_5 + v_5 * (t_6 - t_5) + a_5 \frac{(t_6 - t_5)^2}{2} \tag{28}$$

$[t_6, t_7) \Rightarrow$

$$j(t - t_6) = J \tag{29}$$

$$a(t - t_6) = a_6 + J * (t - t_6);\ a_7 = a_6 + J * (t_7 - t_6) = 0 \tag{30}$$

$$v(t - t_6) = v_6 + a_6 * (t - t_6) + J * \frac{(t - t_6)^2}{2};\ \ v_7 = v_7 + a_6 * (t_7 - t_6) - J * \frac{(t_7 - t_6)^2}{2} = 0 (31)$$

$$p(t-t_6) = p_6 + v_6 * (t-t_6) + a_6 * \frac{(t-t_6)^2}{2} + J * \frac{(t-t_6)^3}{6};$$

$$p_7 = s = p_6 + v_6 * (t_7-t_6) + a_6 * \frac{(t_7-t_6)^2}{2} + J * \frac{(t_7-t_6)^3}{6} \tag{32}$$

The values of the times corresponding to the phases of the movement are:

$$t_1 = t_j \tag{33}$$
$$t_2 = t_a \tag{34}$$
$$t_3 = t_j + t_a \tag{35}$$
$$t_4 = t_v \tag{36}$$
$$t_5 = t_j + t_v \tag{37}$$
$$t_6 = t_v + t_a \tag{38}$$
$$t_7 = t_v + t_a + t_j \tag{39}$$

3. Determining the optimal movement profile and duration of movement phases

Fig. 2 shows the motion curves in the situation where the maximum values for acceleration and speed are not maintained; this implies, see fig. 1,

$$t_1 = t_2, t_3 = t_4 = \frac{t_7}{2}, t_5 = t_6 \tag{40}$$

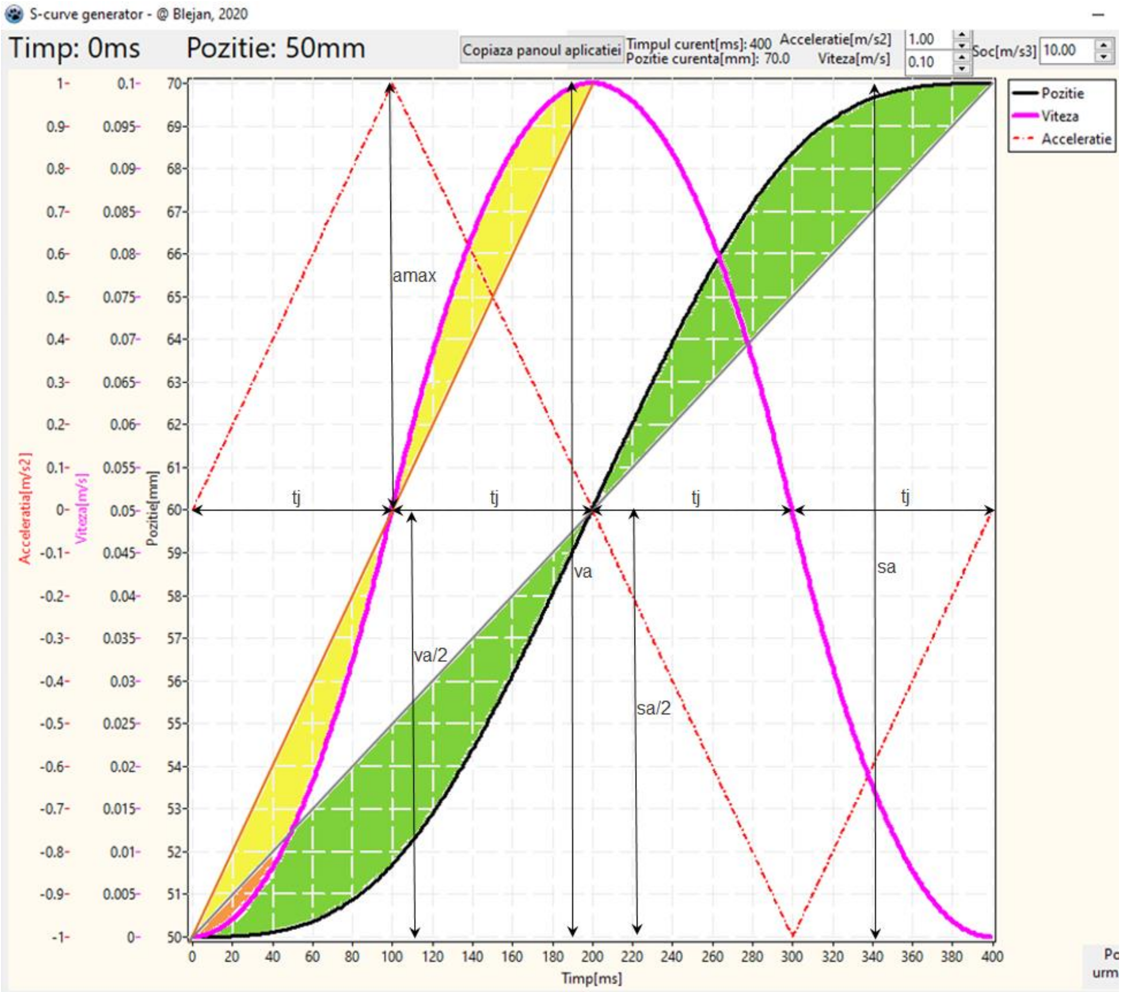

Fig. 2. Movement without acceleration and constant speed

It is observed on the graph that the convex part and the concave part of the velocity and position graphs cancel each other out if you want to calculate the integrals of these quantities.

Thus, the integral of the velocity, at half the duration of the motion, is equal to the area below the graph, which means, according to the previous observation, that the yellow area on the concave side is equal to the yellow area on the convex side of the graph (half stroke) multiplying the value of the movement time and the amplitude of the speed, respectively

$$Integral\ of\ the\ speed = \frac{t_7}{2} * (v_{max} - 0) * \frac{1}{2} = v_{max} * \frac{t_7}{4} \quad (41)$$

Applying a similar reasoning to the position graph, it results

$$Integral\ of\ the\ position = \frac{t_7}{2} * s \quad (42)$$

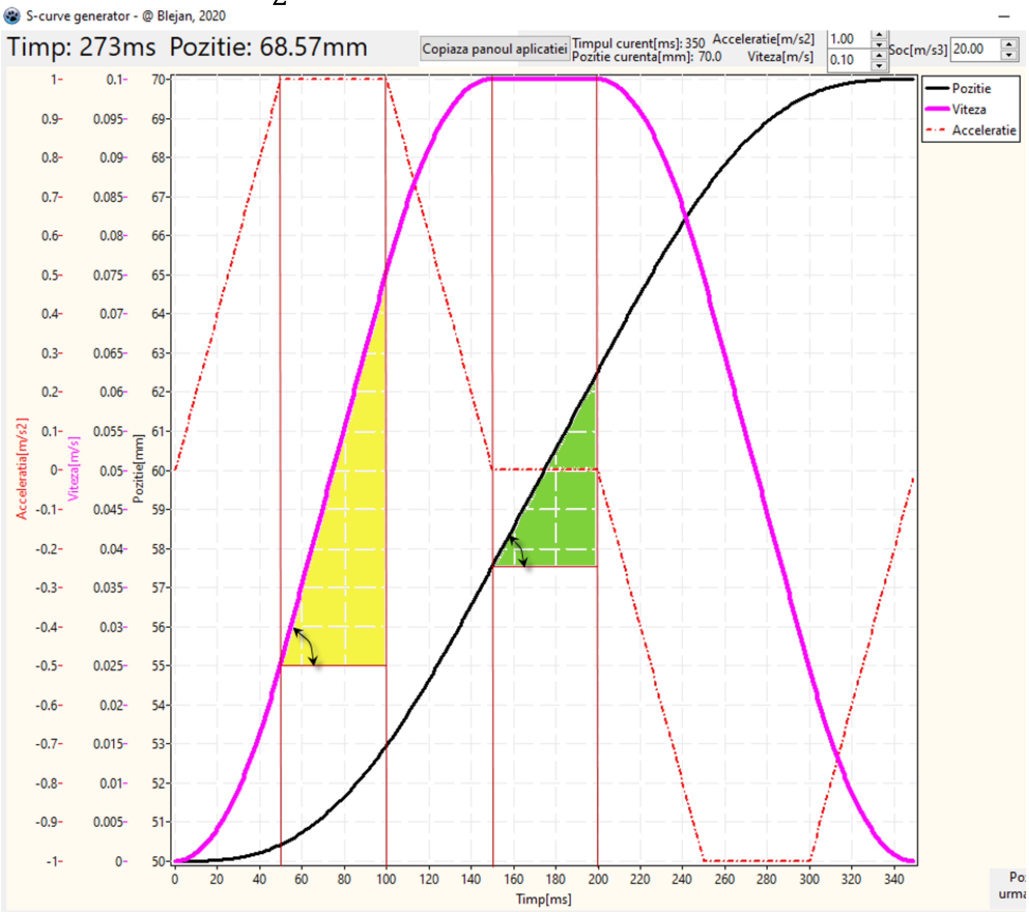

Fig. 3. Motion with acceleration and constant speed

Fig. 3 shows the motion curves in the situation where the maximum values for acceleration and speed are reached and the motion occurs by keeping these values at the maximum limit, the interval $[t_1, t_2]$ in which the motion is performed with the acceleration a_{max}, and in the interval $[t_3, t_4]$ the movement is performed with the speed v_{max}. The corresponding integrals of velocity and position on the mentioned intervals are the areas of the colored triangles, see fig. 3, with yellow for full speed and green for full position, respectively

$$Integral\ of\ the\ speed = \frac{a_{max}}{2} * (t_2 - t_1)^2 \quad (43)$$

$$Integral\ of\ the\ position = \frac{v_{max}}{2} * (t_4 - t_3)^2 \quad (44)$$

because the tangent of the angle marked in the yellow triangle is a_{max}, respectively v_{max} for the angle marked in the green triangle.

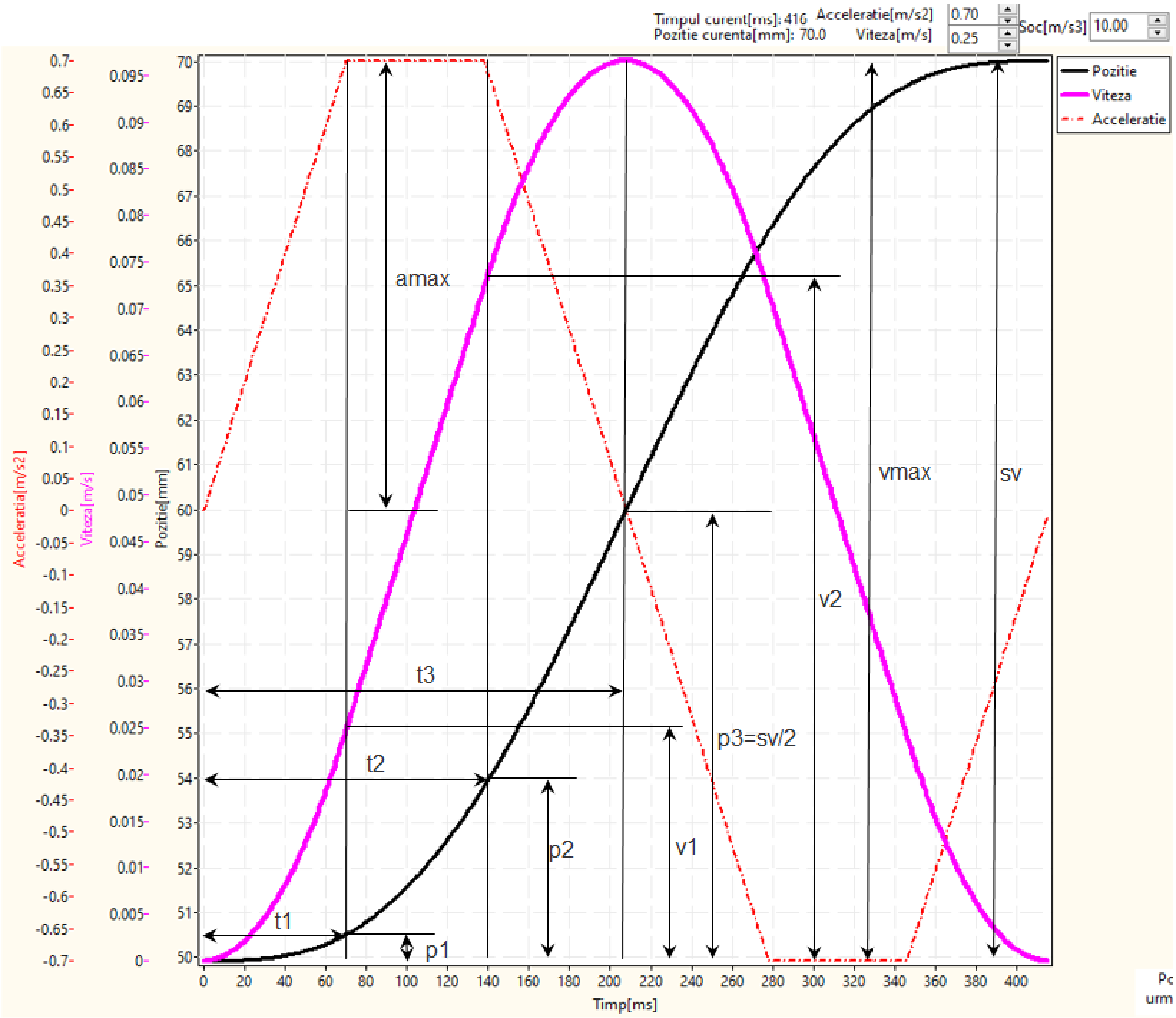

Fig. 4. Motion with constant acceleration

3.1 The definition and calculation of v_a, s_a and s_v

The parameters v_a, s_a and s_v are limit values for speed and space being defined and calculated below.

Considering realization of the movement having as limitation only the maximum value of the acceleration, see fig. 2, will be defined v_a as the maximum value of the speed reached during the movement, and s_a as the value of the stroke at the end of the movement. Reaching the maximum value of the acceleration is followed by stopping the movement as quickly as possible.

To calculate the motion parameters at time t_j, we apply formulas (5), (6), (7) and (8)

$$a_{max} = J * t_j \Rightarrow t_j = \frac{a_{max}}{J} \tag{45}$$

$$\frac{v_a}{2} = J * \frac{t_j^2}{2} \Rightarrow v_a = \frac{a_{max}^2}{J} \tag{46}$$

$$p_{12} = J * \frac{t_j^3}{6} = \frac{a_{max}^3}{6 * J^2} \tag{47}$$

To calculate s_a apply formula (16) to moment $2*t_j$, with the value of t_j given by (45)

$$\frac{s_a}{2} = p(2 * t_j - t_j) = \frac{a_{max}^3}{6 * J^2} + \frac{a_{max}^2}{2 * J} * t_j + a_{max} * \frac{t_j^2}{2} - J * \frac{t_j^3}{6} = \frac{a_{max}^3}{J^2} \Rightarrow s_a = \frac{2 * a_{max}^3}{J^2} \quad (48)$$

Similarly, considering the limitation of the maximum value of the speed, not taking into account the limitation of the value of the position (jump), reaching the maximum value of the speed being followed by stopping the movement as fast as possible, results s_v as the value of the stroke at the end of the movement. We distinguish two cases:
Case 1, a trajectory with the shape of fig. 2, in which the acceleration has a triangular shape ($t_2 = t_1$ and $t_5 = t_6$) and the positive ramp is connected with the negative acceleration ramp
Case 2, a trajectory with the shape of fig. 4, in which the acceleration has a trapezoidal shape and the positive ramp is connected with the negative acceleration ramp ($t_3 = t_4$) and reaching the maximum speed is immediately followed by the decrease of its value in order to stop the movement as fast as possible.
The detection between the two presented cases is made according to the value of the acceleration at time t_2, in the sense it does not reach the maximum imposed value, case 1 or reaches the maximum imposed value and maintains it, case 2. According to formula (46) the choice of case 1 is made if

$$v_{max} * J < a_{max}^2 \Rightarrow \mathit{Case}\ 1, fig.2 \quad (49)$$

, respectively

$$v_{max} * J \geq a_{max}^2 \Rightarrow \mathit{Case}\ 2, fig.4\ \mathit{with\ positive\ and\ negative\ ramp\ of\ acceleration\ connected} \quad (50)$$

We determine the value s_v in case 1, for the beginning we apply the formula (7) with $v_1 = v_{max} / 2$, it results

$$t_j = \sqrt{\frac{v_{max}}{J}} \quad (51)$$

It follows from (6) and (51)

$$a_1 = J * \sqrt{\frac{v_{max}}{J}} \quad (52)$$

, and from (8) and (51) we have

$$p_1 = v_{max} * \sqrt{\frac{v_{max}}{J}} \quad (53)$$

Taking into account that $p_3 = s_v / 2$ and replacing in (16) the values calculated in (51), (52) and (53) results

$$s_v = 2 * v_{max} * \sqrt{\frac{v_{max}}{J}}\ , pentru\ \ v_{max} * J < a_{max}^2 \quad (54)$$

To determine the value s_v in case 2, apply (6) with $a_1 = a_{max}$ obtaining

$$t_1 = \frac{a_{max}}{J} \quad (55)$$

, and then from (7) and (55) we have

$$v_1 = \frac{a_{max}^2}{2 * J} \quad (56)$$

And from (8), (55) it results

$$p_1 = \frac{a_{max}^3}{6 * J^2} \quad (57)$$

Similarly, for the interval t_1 .. t_2, we have

$$t_2 = \frac{v_{max}}{a_{max}} \quad (58)$$

$$v_2 = v_{max} - \frac{a_{max}^2}{2 * J} \tag{59}$$

$$p_2 = \frac{v_{max}^2}{2 * a_{max}} + \frac{a_{max}^3}{6 * J^2} - \frac{a_{max} * v_{max}}{J} \tag{60}$$

For the interval t_2 .. t_3, taking into account that $p_3 = s_v / 2$, it also results

$$s_v = v_{max} * \left(\frac{v_{max}}{a_{max}} + \frac{a_{max}}{J}\right), pentru\ \ v_{max} * J \geq a_{max}^2 \tag{61}$$

3.2 Possible movement profiles

The values of the parameters v_a and s_a represent the maximum limits of the speed and the jump of the position that can be reached, on an S-curve trajectory, without the need for motion with constant acceleration or, in other words, the shape of the acceleration curve to be triangular, see fig. 2.

The value of the parameter s_v represents the maximum limit of the position jump that can be reached without the need for constant speed movement, ie its speed characteristic has a saturation portion, see fig. 4.

Thus, depending on the values of the independent parameters, imposed by the user J, a_{max}, v_{max}, s and the values of the limit parameters of the motion v_a, s_a and s_v we distinguish the following scenarios and below motion scenarios:

A. $v_{max} < v_a; s > s_a$ → the acceleration curve is triangular, the velocity curve is limited so the positive and negative parts of the acceleration are not continuous, fig. 5.A. The final position can be reached if a portion of the trajectory is traveled with constant speed v_{max}.

B. $v_{max} > v_a; s < s_a$ → the acceleration curve is triangular, the velocity curve is not limited so the positive and negative parts of the acceleration are continuous, fig. 5.B

C. $v_{max} < v_a; s < s_a$ → the acceleration curve is triangular, and if $s > s_v$ the maximum speed is reached in case the positive and negative parts of the acceleration are not continuous, fig. 5.C.1, while if $s < s_v$ the positive and negative parts of the acceleration are continuous, fig. 5.C.2. In this case the speed curve is saturated at the value v_{max}, and if the final position $s < s_v$ is not reached, it is necessary to introduce a constant walking portion with the maximum speed.

D. $v_{max} > v_a; s > s_a$ → the acceleration curve is trapezoidal, and if $s > s_v$ the maximum speed is reached in case the positive and negative parts of the acceleration are not continuous, fig. 5.D.1, while if $s < s_v$ the positive and negative parts of the acceleration are continuous, fig. 5.D.2. In this situation, the final position cannot be reached unless a continuously accelerated driving section with the maximum value is entered; the walking portion with constant speed is given by the value of the parameter s_v.

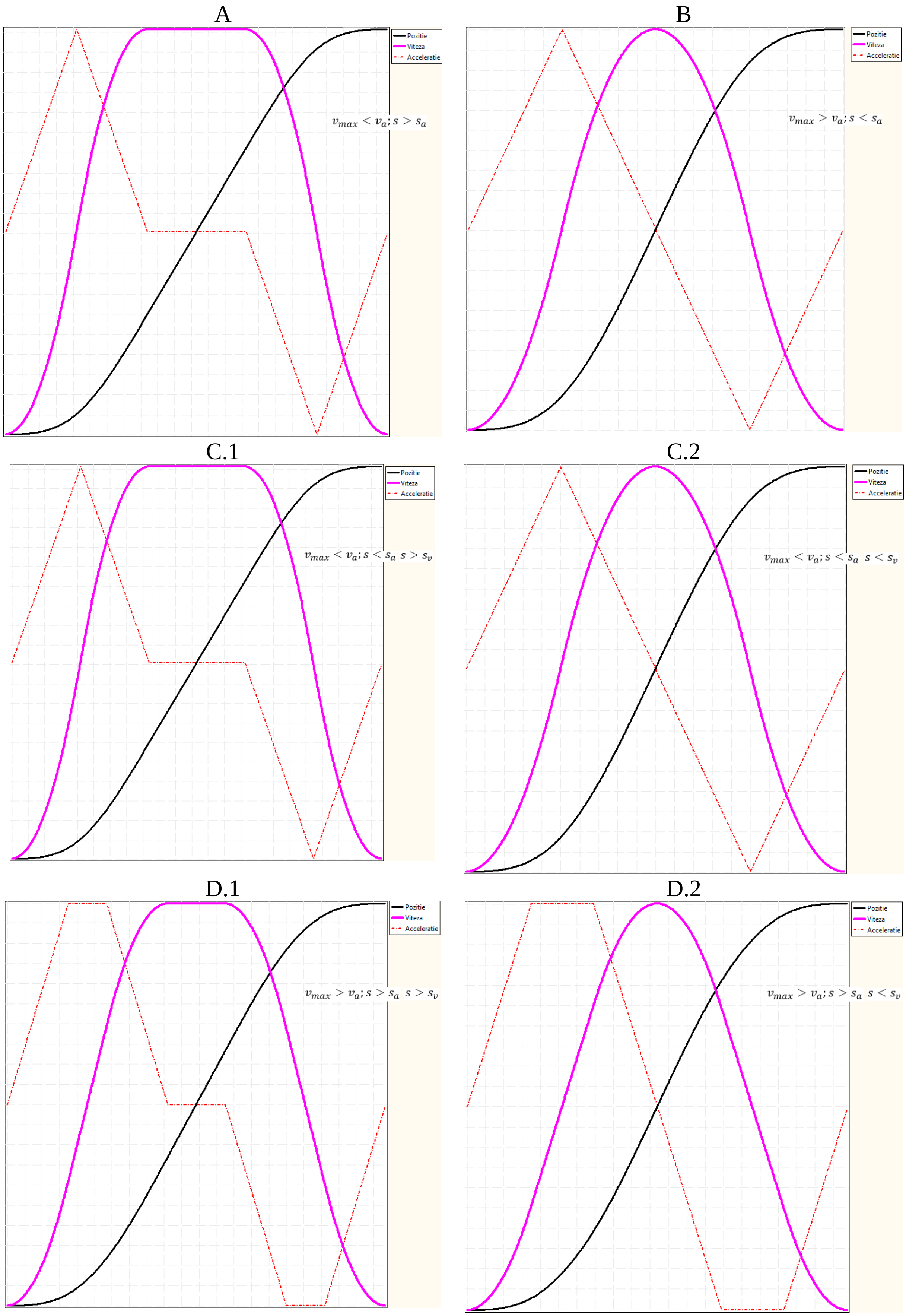

Fig. 5. Possible motion profiles

3.3 Calculation of times corresponding to the phases of movement

To generate the motion profiles, it is necessary to know the values $t_1 \dots t_7$. According to formulas (33)… (39), these values can be calculated if the values t_v, t_a, t_j, defined in fig. 1. The calculation of these times will be performed separately for the six possible profiles presented in fig. 5.

It is observed that at profiles B and C.2 the values of maximum speed and acceleration are not reached. Applying (6), (7), (8) and considering that $t_1 = t_j$, we obtain

$$a(t) = J * t \Rightarrow a(t_j) = J * t_j \tag{62}$$

$$v(t) = J * \frac{t^2}{2} \Rightarrow v(t_j) = J * \frac{{t_j}^2}{2} \tag{63}$$

$$p(t) = J * \frac{t^3}{6} \Rightarrow p(t_j) = J * \frac{{t_j}^3}{6} \tag{64}$$

Since $t_1 = t_2 = t_j$ and $t_3 = 2 * t_j$, replacing in (16) we have

$$p(t - t_j) = J * \frac{{t_j}^3}{6} + J * \frac{{t_j}^2}{2} * (t - t_j) + J * t_j * \frac{(t - t_j)^2}{2} - J * \frac{(t - t_j)^3}{6}$$

$$\Rightarrow p(2 * t_j - t_j) = \frac{s}{2} = J * {t_j}^3 \tag{65}$$

so

$$t_j = \sqrt[3]{\frac{s}{2 * J}} \tag{66}$$

The values t_v, t_a, see fig. 1, fig. 5.B and fig. 5.C.2, are

$$t_a = t_j \tag{67}$$

$$t_v = 2 * t_j \tag{68}$$

Trajectories A and C.1 reach the value of the maximum speed. The equations of motion will be applied at intervals.

Interval $0 \,..\, t_j$: apply (6), (7) and (8) for the calculation of the initial values on the next interval.

Interval $t_j \,..\, 2 * t_j$:

$$a(t - t_j) = J * t_j - J * (t - t_j) \tag{69}$$

$$v(t - t_j) = J * \frac{t_j^2}{2} + J * t_j * (t - t_j) - J * \frac{(t - t_j)^2}{2} \tag{70}$$

$$p(t - t_j) = J * \frac{t_j^3}{6} + J * \frac{t_j^2}{2} * (t - t_j) + J * t_j * \frac{(t - t_j)^2}{2} - J * \frac{(t - t_j)^3}{6} \tag{71}$$

Substituting in (70) $t = 2 * t_j$ and taking into account that $v(2 * t_j) = v_{max}$ at the moment $2 * t_j$

$$\Rightarrow t_j = \sqrt{\frac{v_{max}}{J}} \tag{72}$$

, and from (71) and (72) with $t = 2 * t_j$, also at the moment $2 * t_j$

$$\Rightarrow p_{2tj} = v_{max} * \sqrt{\frac{v_{max}}{J}} \tag{73}$$

The interval $2 * t_j \,..\, 2 * t_j + \frac{t_v - 2 * t_j}{2}$, ie half of the total duration of the movement, when the distance $\frac{s}{2}$ has been covered:

replacing in (4), with the initial values calculated in (69), (70) and (71), we obtain

$$p(t - 2 * t_j) = v_{max} * \sqrt{\frac{v_{max}}{J}} + v_{max} * (t - 2 * t_j) \tag{74}$$

, and from (74) at the moment $t = 2 * t_j + \frac{t_v - 2*t_j}{2}$ the space travelled is $\frac{s}{2}$

$$\Rightarrow t_v = \frac{s}{v_{max}} \tag{75}$$

Because the acceleration trajectory is triangular, we have

$$t_a = t_j \tag{76}$$

In the case of trajectory D.1, the values of maximum speed and acceleration are reached. Taking into account the explanations regarding fig. 2 and fig. 3 and relations (41), (42), (43) and (44), we have

- The value of the shock integral on the interval *0 ..* t_1 is the value of the acceleration at the moment $t_1 = t_j$, which has the value a_{max}. It is thus obtained

$$t_j = \frac{a_{max}}{J} \tag{77}$$

- The value of the acceleration integral on the interval *0 ..* t_3 is the value of the speed at time t_3, ie v_{max}. So

$$t_a = \frac{v_{max}}{a_{max}} \tag{78}$$

Similarly, the value of the velocity integral over the interval *0 ..* t_7 is the value of the position at time t, i.e. s. So

$$t_v = \frac{s}{v_{max}} \tag{79}$$

Trajectory D.2, see fig. 6, reaches and maintains the value of the maximum acceleration, and the value of the maximum speed is not reached.

Interval *0 ..* t_j : apply (6), (7) and (8) for the calculation of the initial values on the next interval.

Because $a(t_j) = a_{max}$ and taking into account (5) we have

$$t_j = \frac{a_{max}}{J} \tag{80}$$

Interval t_j .. t_a :

$$a(t - t_j) = a_{max} \tag{81}$$

$$v(t - t_j) = J * \frac{t_j^2}{2} + a_{max} * (t - t_j) \tag{82}$$

$$p(t - t_j) = J * \frac{t_j^3}{6} + J * \frac{t_j^2}{2} * (t - t_j) + a_{max} * \frac{(t - t_j)^2}{2} \tag{83}$$

Substituting in (82) $t = t_j + \frac{t_a - t_j}{2}$, when the speed reaches half of the maximum value

$$\frac{v_{tv}}{2} = J * \frac{t_j^2}{2} + a_{max} * \frac{t_a - t_j}{2} \tag{84}$$

Substituting t_j from (80) into (84) results

$$v_{tv} = a_{max} * t_a \tag{85}$$

Taking into account that *Space = Integral of velocity*, considering the moment $t_v = t_a + t_j$ and taking into account fig. 2 and fig. 3 in which it is shown that the part of the areas of the convex and concave part of the graph is cancelled, see fig.6 where the integral is marked in yellow, we have

$$\frac{s}{2} = \frac{t_v}{2} * 2 * v_{tv} \tag{86}$$

Substituting $t_v = t_a + t_j$ and (85) in relation (86) we obtain

$$s = 2 * a_{max} * t_a * \left(t_a + \frac{a_{max}}{J}\right) \tag{87}$$

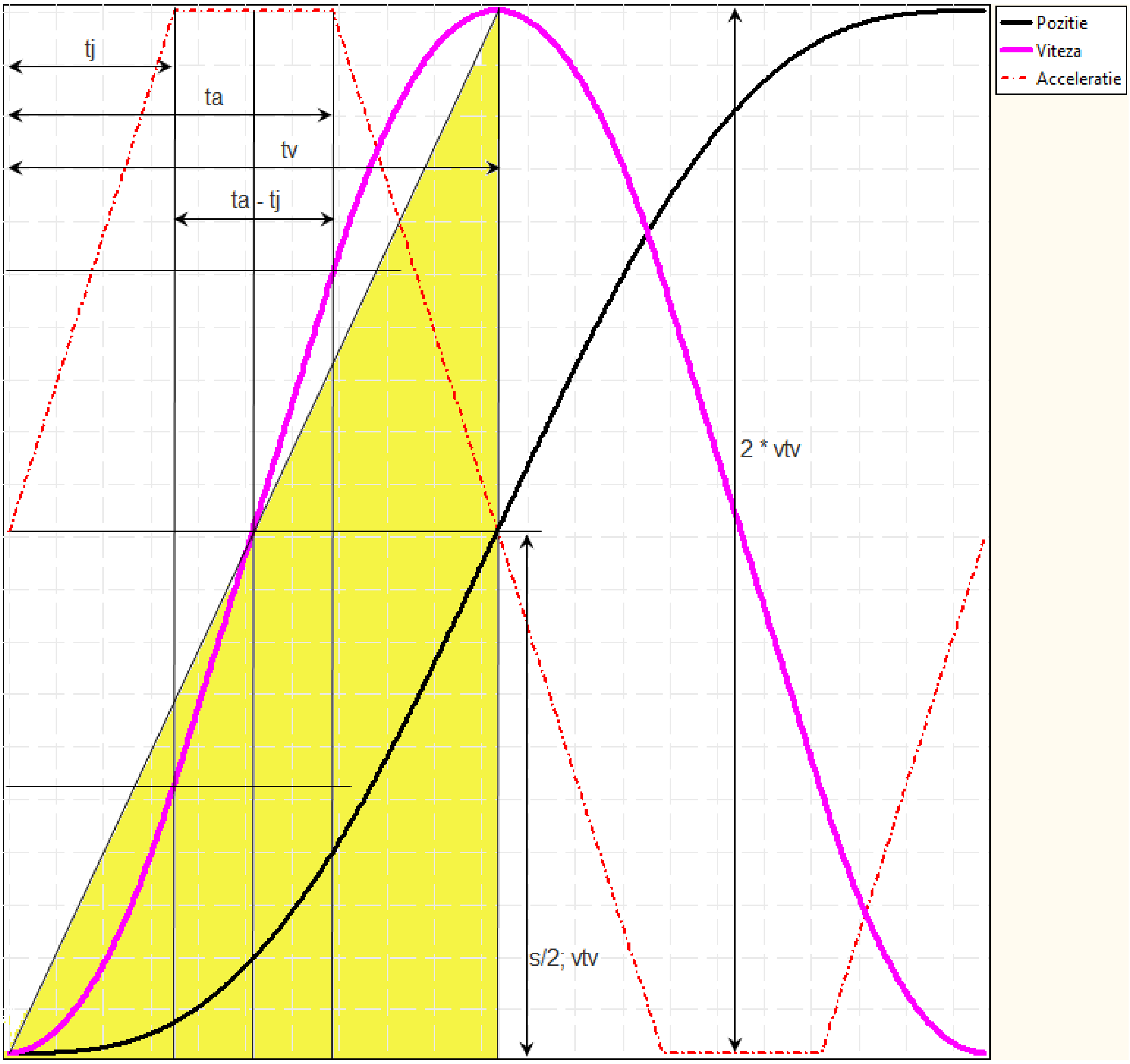

Fig. 6. Case D.2 - the value of the maximum acceleration is reached, the value of the maximum speed is not reached

Solving the equation of degree 2 (87) and taking the positive solution, we have

$$t_a = \frac{1}{2} * \left(\sqrt{\frac{4 * s * J^2 + a_{max}^3}{a_{max} * J^2}} - \frac{a_{max}}{J} \right) \tag{88}$$

and

$$t_v = t_a + t_j \tag{89}$$

4. Real-time motion profile generation algorithm

The generation of S-curve motion profiles for position, speed, acceleration and shock is triggered when a significant jump occurs at the generator input followed by the calculation of the current values of position, speed, acceleration and shock, values that are delivered at the output of the S-curve generator in real time. At each change of the current time, it is necessary to call the calculation of the new values of position, speed, acceleration and shock corresponding to the current value of time; the generation of the profiles is completed when the final position is reached, the value of which is the algebraic sum between the value of the initial position and the value of the position jump.

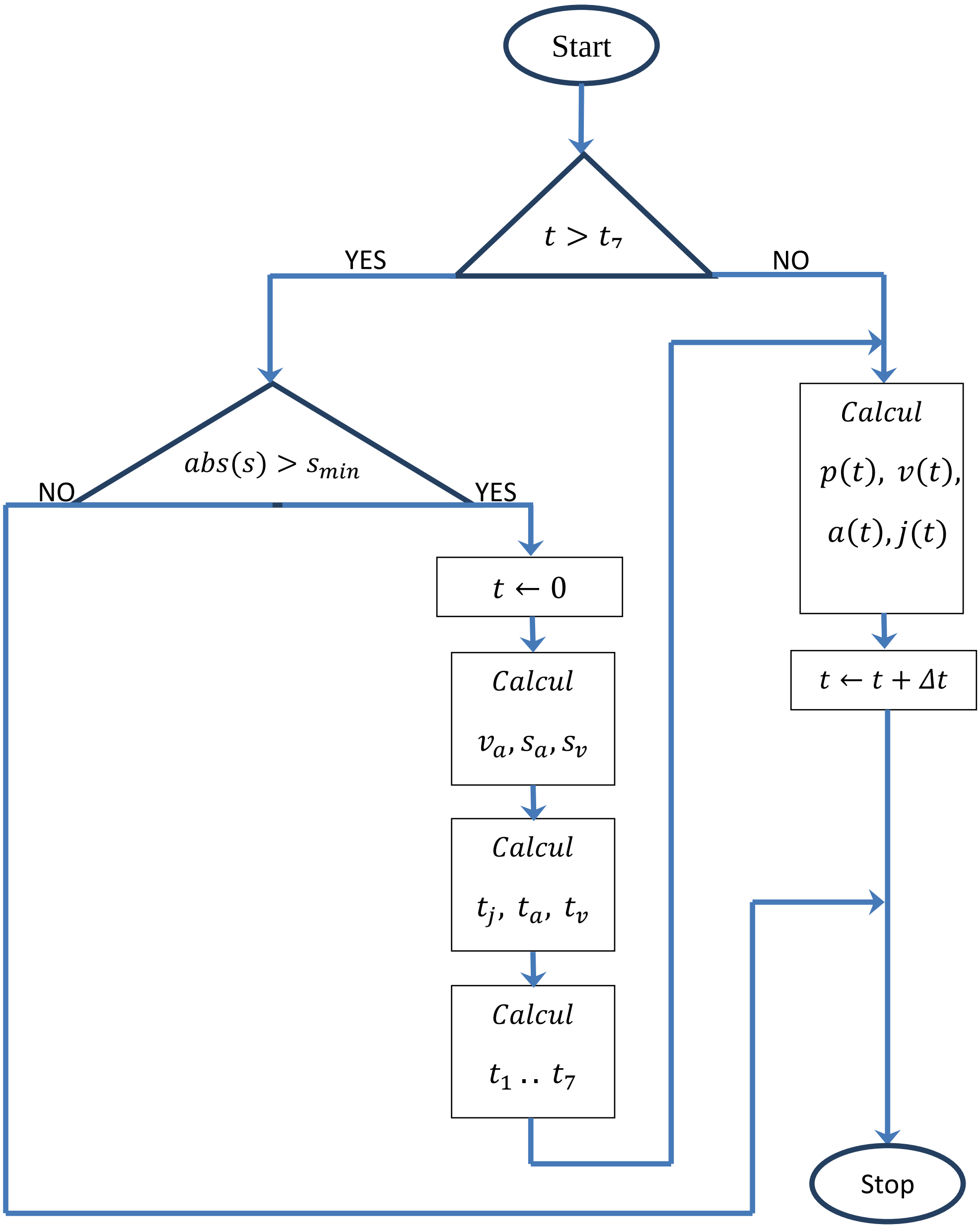

Fig. 7. Real-time S-curve generation algorithm

Fig. 7 presents the S-curve generation algorithm in real time, and further the calculation steps will be presented, in correlation with the previously presented formulas.
Thus *Calcul* v_a, s_a, s_v is performed according to (46), (48) for v_a, s_a respectively (54) or (61) for s_v, depending on where we fit. The calculated values allow us to establish one of the motion trajectories shown in fig. 5, which allows us to calculate the values t_j, t_a, t_v thus

- For trajectories B and C.2 apply (66), (67) and (68)
- For trajectories A and C.1 (72), (75) and (76) apply
- For trajectory D.1 apply (77), (78) and (79)
- For trajectory D.2 apply (80), (88) and (89)

The calculation of the values t_1 .. t_7 is presented in the relations (33) .. (39) and uses the values t_j, t_a, t_v calculated previously.

Finally, the values *p(t), v(t0), a(t)* and *j(t)* are calculated using the relations (1), (2), (3) and (4) in which the initial values p_0, v_0 and a_0 are zero at the first calculation step (*t = 0*), and then are the values *p(t), v(t0), a(t)* and *j(t)* at the previous calculation step. The values of *J* are those corresponding to the current time, on the current trajectory, see fig. 5; that is, one of the values *J, -J* or *0*.

The value of the parameter *Δt* is the time interval between two successive calls of the algorithm shown in fig. 7, in other words the integration step used to calculate the values of *p(t), v(t0)* and *a(t)*.

5. Confirmation of the proposed algorithm by simulation

The units of measurement for time and space used in the simulation will be *ms* and *mm*, and we need to use the units of measurement that use the following conversion ratios:

$$1\,m = 1000\,mm \tag{90}$$

$$1\,\frac{m}{s} = 1\,\frac{mm}{ms} \tag{91}$$

$$1\,\frac{m}{s^2} = 10^{-3}\,\frac{mm}{ms^2} \tag{92}$$

$$1\,\frac{m}{s^3} = 10^{-6}\,\frac{mm}{ms^3} \tag{93}$$

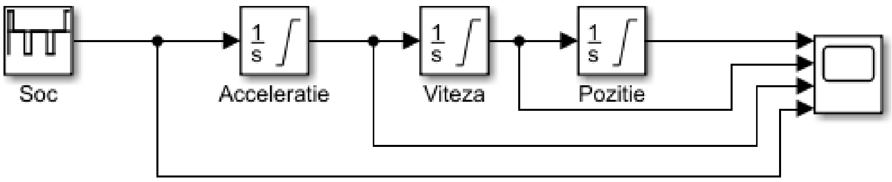

Fig. 8. Simulation model used

The simulation model used, see fig. 8, contains three "Integrator" blocks (*Acceleration, Speed* and *Position*) with saturation, the saturation limits being $\pm a_{max}, \pm v_{max}$ and *[0 .. 100 mm]*. The jerk is generated with the "Repeating Sequence Stair" block, and the output signals are viewed with the "Scope" block.

The S-curve generator was implemented according to the algorithm shown in fig. 7, in the Lazarus application development environment (www.lazarus-ide.org) under Windows. 32-bit arithmetic was used, in floating point with 24-bit mantissa and 8-bit exponent, respectively single type variables, see https://wiki.freepascal.org/IEEE_754_formats.

The evaluation of the developed algorithm was performed by programming the jerk generator from the simulation model, fig. 8, with the jerk route data from the S-curve generator, respectively the jerk values correlated with the movement times. The data were packaged for simulation as a vector with a time resolution of *1 ms*. Because of this it was necessary to convert the units of measurement according to relations (90), (91), (92) and (93).

The assessment criterion was the value of the position error, calculated as the difference between the values of the final positions of the S-curve generator and of the simulation on the model in fig. 8.

The simulation was performed in MATLAB & Simulink, variant R2019b.

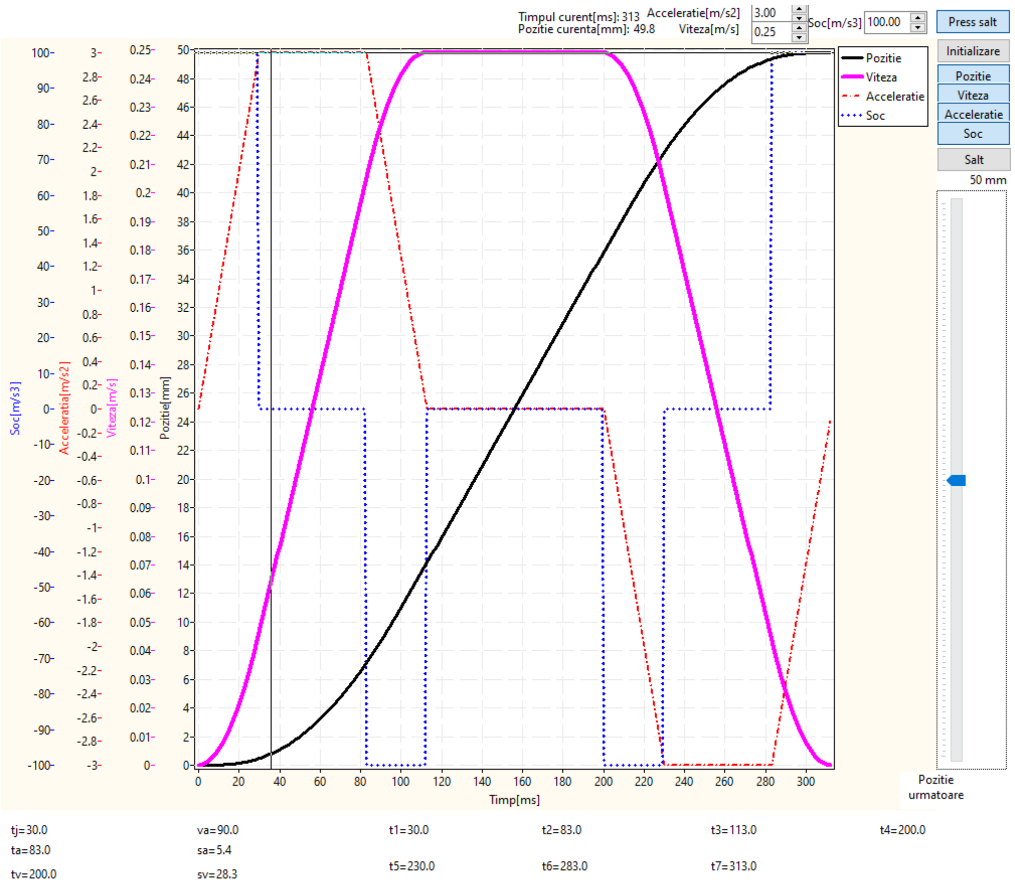

Fig. 9. Movement profile D.1

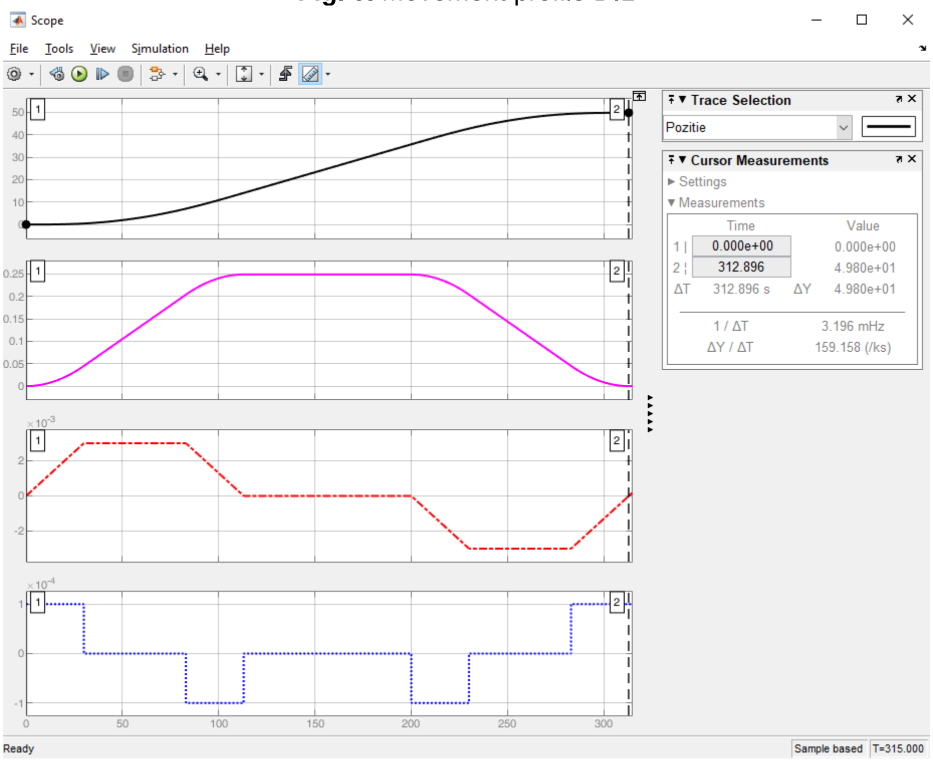

Fig. 10. Simulation results for case D.1

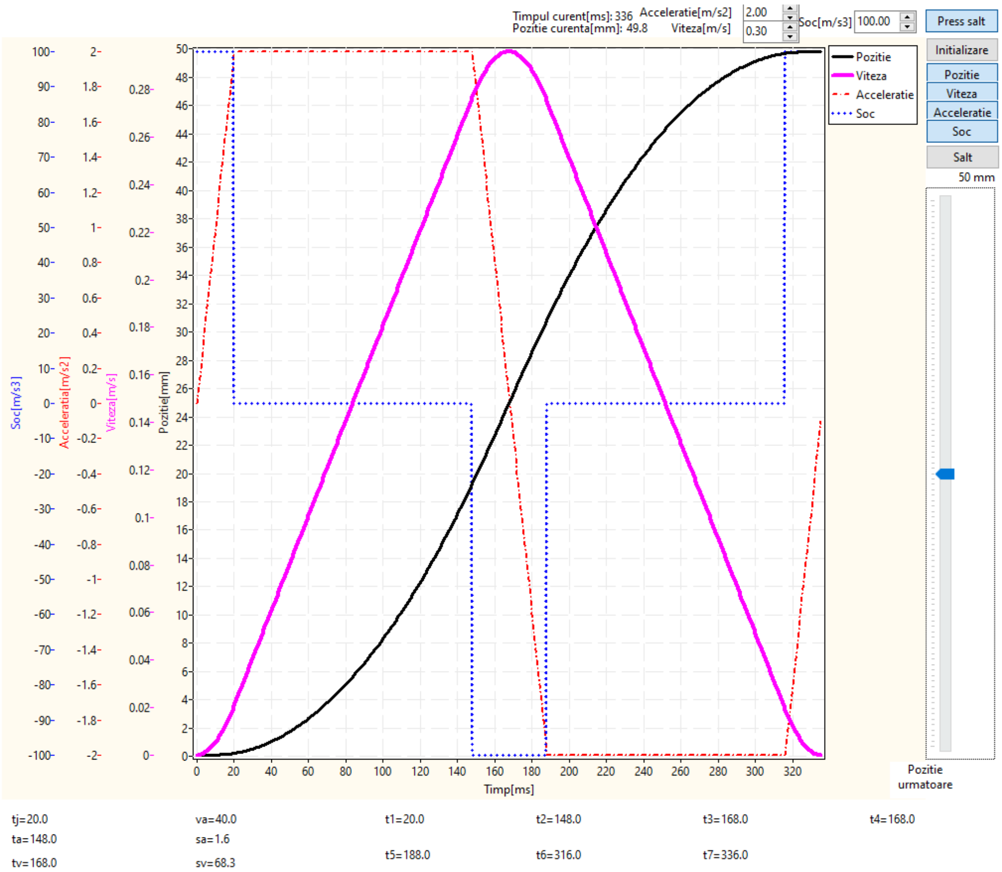

Fig. 11. Movement profile D.2

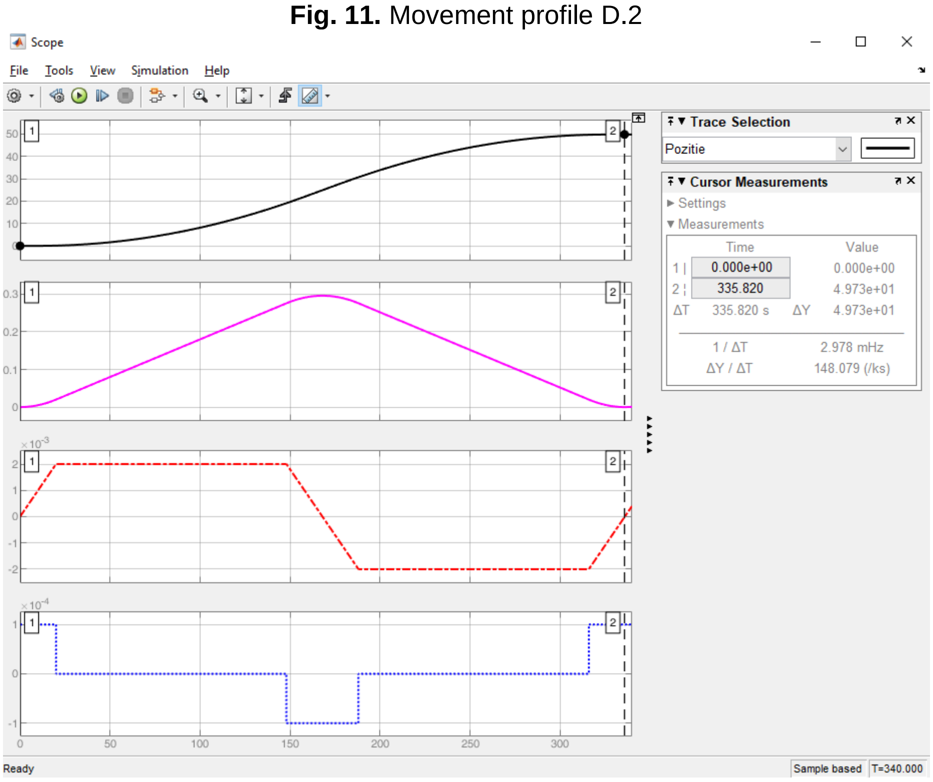

Fig. 12. Simulation results for case D.2

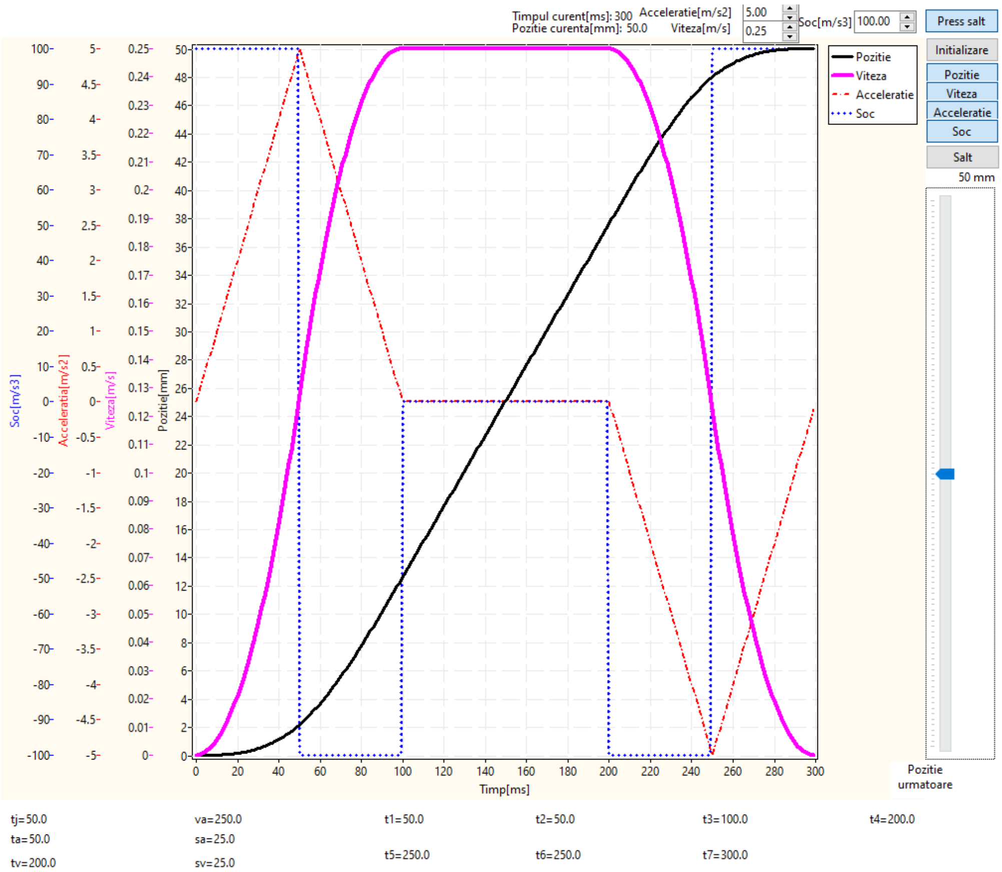

Fig. 13. Movement profiles A and C.1

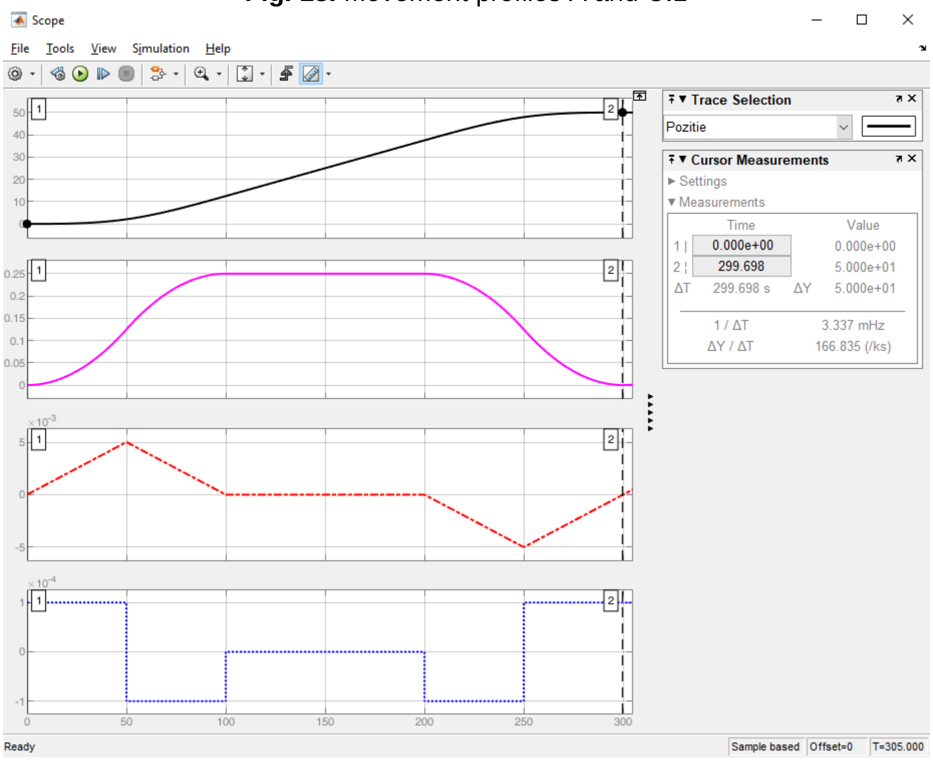

Fig. 14. Simulation results for cases A and C.1

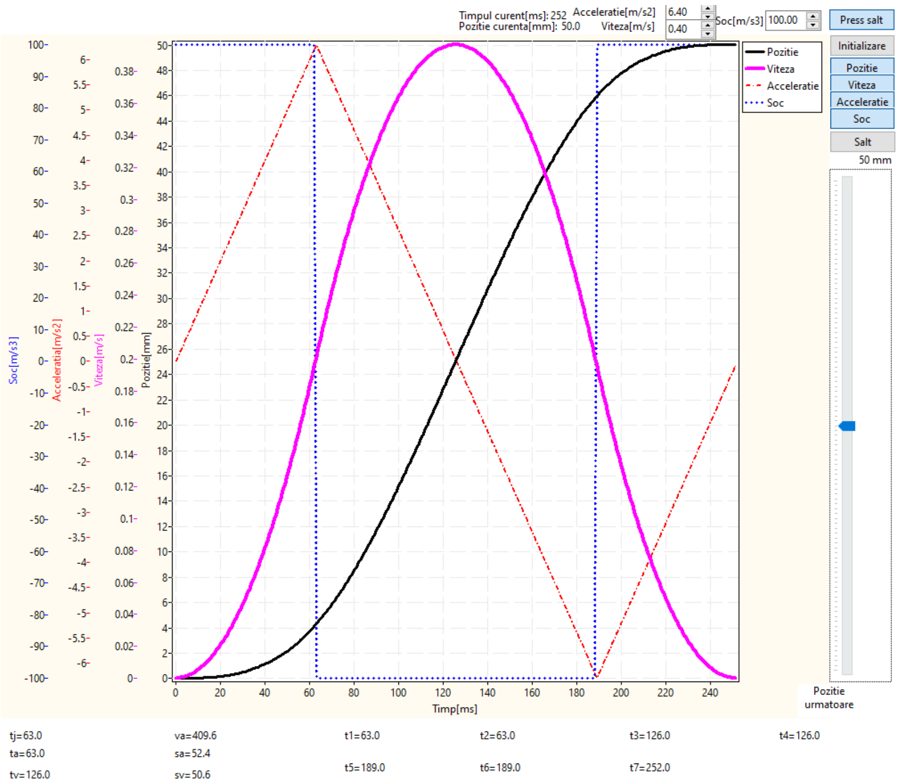

Fig. 15. Movement profiles B and C.2

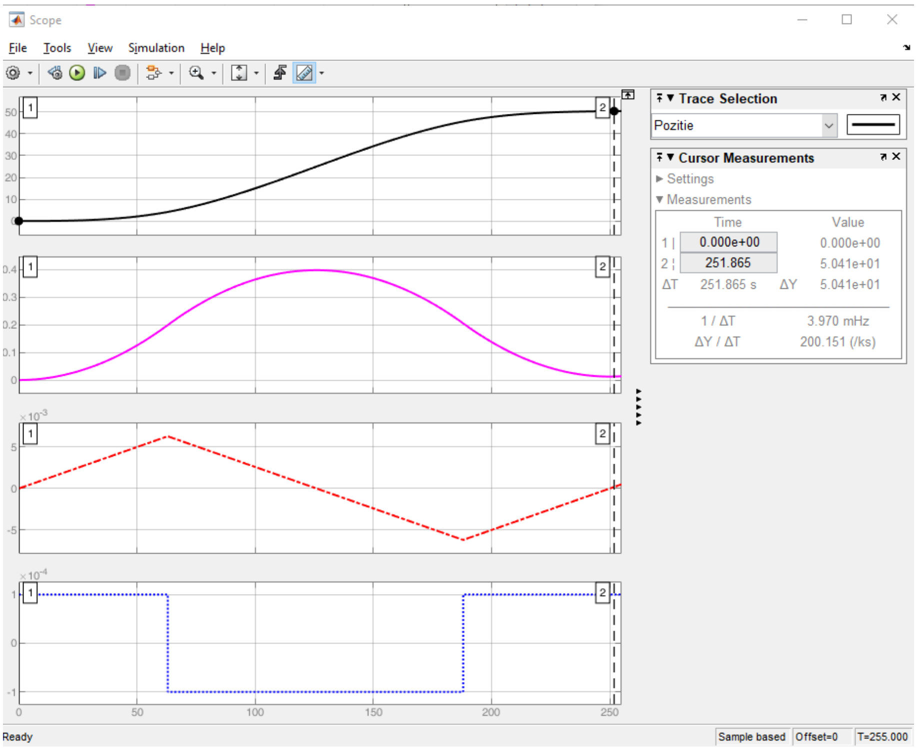

Fig. 16. Simulation results for cases B and C.2

Fig. 9 shows the motion profiles generated for a *50mm* jump, with limitations

$$J = 100\frac{m}{s^3} = 10^{-4}\frac{mm}{ms^3}\ ; a_{max} = 3\frac{m}{s^2} = 3 * 10^{-3}\frac{mm}{ms^2}\ ;\ v_{max} = 0.25\frac{m}{s} = 0.25\frac{mm}{ms}$$

, and the signal for the simulated jerk, see fig. 10, is the vector

[1e-4+zeros(1,30) zeros(1,53) -1e-4+zeros(1,30) zeros(1,87) -1e-4+zeros(1,30) zeros(1,53) 1e-4+zeros(1,30)].
In simulation the final value of the position is *49.8mm*, resulting in a position error that has the value of *0.2mm*, or in percent $\frac{0.2}{50} * 100 = 0.4\%$.
Fig. 11 shows the motion profiles generated for a *50mm* jump, with limitations

$$J = 100\frac{m}{s^3} = 10^{-4}\frac{mm}{ms^3}\,; a_{max} = 2\frac{m}{s^2} = 2 * 10^{-3}\frac{mm}{ms^2}\,;\ v_{max} = 0.3\frac{m}{s} = 0.3\frac{mm}{ms}$$

, and the signal for the simulated jerk, see fig. 12, is the vector
[1e-4+zeros(1,20) zeros(1,128) -1e-4+zeros(1,40) zeros(1,128) 1e-4+zeros(1,20)].
In simulation the final value of the position is *49.73mm*, resulting in a position error that has the value of *0.27mm*, or in percent $\frac{0.27}{50} * 100 = 0.54\%$.
Fig. 13 shows the motion profiles generated for a *50mm* jump, with limitations

$$J = 100\frac{m}{s^3} = 10^{-4}\frac{mm}{ms^3}\,; a_{max} = 5\frac{m}{s^2} = 5 * 10^{-3}\frac{mm}{ms^2}\,;\ v_{max} = 0.25\frac{m}{s} = 0.25\frac{mm}{ms}$$

, and the signal for the simulated jerk, see fig. 14, is the vector
[1e-4+zeros(1,50) -1e-4+zeros(1,50) zeros(1,100) -1e-4+zeros(1,50) 1e-4+zeros(1,50)].
In simulation the final value of the position is *50mm*, resulting in a position error that has the value of *0mm*, or in percent $\frac{0.27}{50} * 100 = 0\%$.
Fig. 15 shows the motion profiles generated for a *50mm* jump, with limitations

$$J = 100\frac{m}{s^3} = 10^{-4}\frac{mm}{ms^3}\,; a_{max} = 6.4\frac{m}{s^2} = 6.4 * 10^{-3}\frac{mm}{ms^2}\,;\ v_{max} = 0.4\frac{m}{s} = 0.4\frac{mm}{ms}$$

, and the signal for the simulated jerk, see fig. 16, is the vector
[1e-4+zeros(1,63) -1e-4+zeros(1,126) 1e-4+zeros(1,63)].
In simulation the final value of the position is *50.41mm*, resulting in a position error that has the value of *0.41mm*, or in percent $\frac{0.41}{50} * 100 = 0.82\%$.

6. Conclusions

After an exhaustive mathematical analysis of the movement between two points following a third order movement profile, a real-time algorithm is developed for the implementation of an online S-curve profile generator.
In order to confirm the correctness of the developed algorithm, a mathematical model of the third order motion is made using the MATLAB simulation environment. The motion profiles generated by the real-time algorithm and by the simulation model were compared in order to evaluate the performances of the S-curve generator. The parameter followed was an error in reaching the final position of the movement. The motion profile generator used 32-bit floating point arithmetic, with a 24-bit mantissa and an 8-bit exponent, as well as a calculation interval of *1ms*. The value of the error of reaching the final position, analysed in all possible cases of movement, turned out to be less than 1% of the value of the position jump.
In order to improve the value of the error, it is proposed to use the representation of numbers with a higher precision in the elaboration of the generator algorithm, for example the 64-bit floating point arithmetic, as well as a reduction of the calculation interval under *1ms*.

References

[1] Lewin, Chuck. "S-Curve Motion Profiles - Vital For Optimizing Machine Performance." https://www.pmdcorp.com/resources/type/articles/get/s-curve-profiles-deep-dive-article. Accessed November 26, 2020.
[2] Mu, Haihua, Yunfei Zhou, Sijie Yan, and Aiguo Han. "Third-order trajectory planning for high accuracy point-to-point motion." *Frontiers of Electrical and Electronic Engineering in China* 4, no.1 (March 2009): 83–87.
[3] ***. "Real-time computing", *Wikipedia*. https://en.wikipedia.org/wiki/Real-time_computing. Accessed November 26, 2020.