Using the Fractional Derivative of Grundal Letnikov to Approximate the Profile of a Rectangular Channel

Dra. **Maritza Liliana ARGANIS JUÁREZ[1*]**, M.Eng. **Margarita PRECIADO JIMÉNEZ[2]**, B.S. Student **Carlos SANTAMARÍA DÍAZ[3]**, Dr. **José Luis HERRERA ALANÍS[4]**, Dr. **José Luis ARAGÓN HERNÁNDEZ[5]**, M.A. Eng. **Jesús Javier CORTÉS ROSAS[5]**, M.A.Eng. **Miguel Eduardo GONZÁLEZ CÁRDENAS[5]**, M.A. Eng. **Víctor Damián PINILLA MORÁN[5]**, Eng. **Alfonso SALAZAR MORENO[5]**

[1] Universidad Nacional Autónoma de México. Instituto de Ingeniería, Av. Universidad 3000 Ciudad Universitaria Edificio 17 Cub.328. Coyoacán CDMX México 04510

[2] Universidad Nacional Autónoma de México. Facultad de Ciencias, Av. Universidad 3000 Ciudad Universitaria Coyoacán CDMX México 04510

[3] Instituto Mexicano de Tecnología del Agua. Blvd. Paseo Cuauhnáhuac 8532, Progreso, 62550 Jiutepec, Mor México

[4] Sistema de Aguas de la Ciudad de México. Calle Nezahualcóyotl Colonia Centro (Área 8), Cuauhtémoc, Ciudad de México. Edificio 109, Piso 4CDMX. 06080

[5] Universidad Nacional Autónoma de México. Facultad de Ingeniería, Av. Universidad 3000 Ciudad Universitaria Edificio 17 Cub.328. Coyoacán CDMX México 04510

* MArganisJ@iingen.unam.mx

Abstract: *A two-term numerical scheme was used from the fractional derivative Grundal Letnikov, considering order a means to find the M-type profile of the water of a rectangular channel. Factors were used that associate the one-half derivative with the one-order derivative as a function of the chain x. The factors obtained from considering polynomial functions of degree 3 allowed to obtain better approximations to the flow profile at some points far from the critical depth.*

Keywords: *Derivative of Grundal Letnikov, gradually varied flow, rectangular channel, fractional calculus*

1. Introduction

The fractional calculus [1] applied in mechanical, physics and engineering problems [2,3,4,5], recently in science health [6]; is a useful and relatively simple tool when combined with numerical analysis [7], in the field of physics fractional calculus has begun to be used to solve diffusive equation problems [8,9]. [10] present factors that relate the behavior of the fractional derivative one-half with the derivative of order one, when simple polynomial functions are used. In this study, the equations that relate the proportion factor of the one-half derivative with the one-order derivative were obtained, as a function of the independent variable x, said factor was used in the result that gave a finite difference scheme fractional based on the fractional derivative of Grundal Letnikov, it was used to solve the equation of gradually varied flow applied to a rectangular laboratory channel, of which measurements and calculations were carried out with the direct passage method. The results obtained are presented by assuming that the derivative one means allows a better approximation of the variation of the flow profile, with respect to the chain x. The previous assumption was only fulfilled in some points far from the control section for the estimation of the M-type profile that the rectangular channel used has.

2. Methodology

According to [8], the first derivative y'(x) of a continuous function y (x) can be approximated with the concept of backward finite differences as

$$y'(x) = \frac{y(x)-y(x-\Delta x)}{\Delta x} \tag{1}$$

If the function is derived again using finite differences backwards, we obtain:

$$y'(x) = \frac{y(x)-2y(x-\Delta x)+y(x-2\Delta x)}{\Delta x^2} \quad (2)$$

By mathematical induction the numerical derivative of order n proposed by [8] results:

$$y^n(x) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} y(x-\Delta x) \quad (3)$$

Where

$$\binom{n}{r} = \frac{n(n-1)(n-2)..(n-r+1)}{r!} \quad (4)$$

[8] proposes that if n is not an integer, the form of the fractional derivative of Grundal Letnikov in finite differences results:

$$y^\gamma(x) = \frac{1}{(\Delta t)^\gamma}\sum_{r=0}^{n}(-1)^r \binom{\gamma}{r} y(x-\Delta x) \quad (5)$$

In the previous equation γ is the order of the fractional derivative that can be a real number. The function is considered defined in a closed interval of values [x_i, x], where $n\Delta x = x - x_i$.

2.1 Fractional finite difference scheme of a term based on the derivative of Grundal Letnikov

Taking into account equation (5), if the value of the function y(i-1) is known and it is desired to estimate its value at an instant y(i), knowing the derived function $f(x,y) = y^\gamma(x)$, with n=1 a scheme similar to Euler's method is obtained, but for fractional finite differences:

$$y(i) = \binom{\gamma}{1} y(i-1) + (\Delta x)^\gamma f(x_i, y_i), i = 2,3, \ldots m \quad (6)$$

In this study a value of γ=1/2 was considered.

2.2 Derivative factor that associates one-half with derivative of order one as a function of x

According to [10] factor $f=y^{1/2}/y'$ was calculated for each point of 2 and 3 degrees polynomials, and it was drawn against the independent variable x, with which following equations were found:

$$f1 = 0.0243x^2 + 0.9795x - 1.9499 \quad (7)$$

$$f2 = -0.001 + 0.116x + 0.7474 \quad (8)$$

Where x is an independent variable x, f1 and f2 are factors that relate mean derivative with order 1 derivative considering 2 and 3 degrees' polynomials, respectively.

2.3 Gradually varying flow equation

Gradually varied flow equation for a prismatic channel is given by [11]:

$$y'(x) = \frac{(S_0 - S_f)}{1 - Fr^2} \quad (9)$$

Where S_0 is dimensionless channel slope bottom, S_f is dimensionless friction slope (in this paper its average value was considered between two successive points and using Manning equation), F_r is Froude number, (also for calculation was taken average value between two successive points).

3. Application and results

Equation 6 scheme was applied considering γ = 1/2 and Δx = 0.5 m, to obtain water profile (Figure 1) of a hydraulics laboratory rectangular channel from Faculty of Engineering, UNAM, with geometric data and hydraulic conditions given in Table 1. Starting from critical condition y_c = 0.12 m, to normal condition y_n = 0.192, slope S_0 = 0.00128 and critical slope S_c = 0.0046, channel profile type was estimated from type M because $y_n > y_c$ and $S_0 < S_c$. For calculation purposes, an initial chain xi = 10 m was used. One-half derivative f (xi, yi) using t equation scheme of 6 was obtained by multiplying order 1 derivative from equation 9 multiplied by f_1 factor or f2 for each interest point from equations 7 and 8.

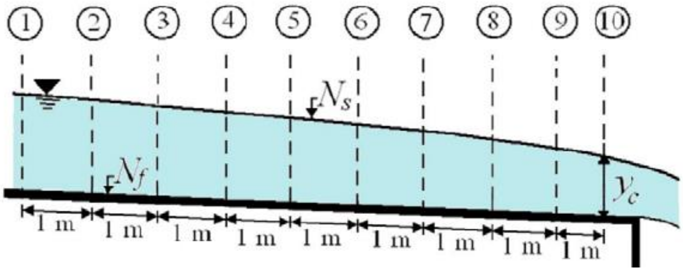

Fig. 1. Water profile to be calculated. Rectangular Chanel (Source: [12])

Table 1. Experimental rectangular channel data

Variable	Value	Units
Q	0.0192	m^3/s
n	0.009	
S0	0.00128	
b	0.2	m
g	9.81	m/s^2

In Table 2 we show x chaining reported at each meter by laboratory measurements were made; the data of the depth measured in laboratory y_{lab}, calculated data with direct step method y_{step}, and depths estimated values with scheme and factors f_1 and f_2, represented by yf_1, 0.5y f_2 and 0.25 f_2; equation 6 the latter correspond to a Δ_x = 0.5m and Δ_x = 0.25 m, respectively. Figure 2 shows graphical comparison of these results.

Table 2. Measured and calculated profile Results from rectangular channel

x Section, m	ylab, m	ystep	y_{f1}, m	0.5y_{f2} ,m	0.25y_{f2} ,m	%errorstep	%errorf1	%error0.5f_2	%error0.5f_2
10	0.109	0.109	0.116	0.116	0.116	0.00	6.4	6.4	6.4
9	0.13	0.137	3.069	0.488	0.08054287	5.38	2260.5	275.3	38.0
8	0.133	0.137	2.125	0.014	0.12029287	3.01	1497.8	89.3	9.6
7	0.139	0.143	1.161	1.008	0.13054361	2.88	735.0	625.3	6.1
6	0.141	0.145	0.034	0.729	0.20255578	2.84	75.7	416.8	43.7
5	0.146	0.149	0.878	0.423	0.12024771	2.05	501.0	189.7	17.6
4	0.15	0.152	0.177	0.049	2.35906407	1.33	18.2	67.6	1472.7
3	0.153	0.155	0.040	0.049	0.14462849	1.31	73.9	68.0	5.5
2	0.155	0.157	0.158	0.034	0.13669956	1.29	1.9	78.3	11.8
1	0.162	0.162	0.220	0.070	43.8272592	0.00	36.1	56.8	26953.9

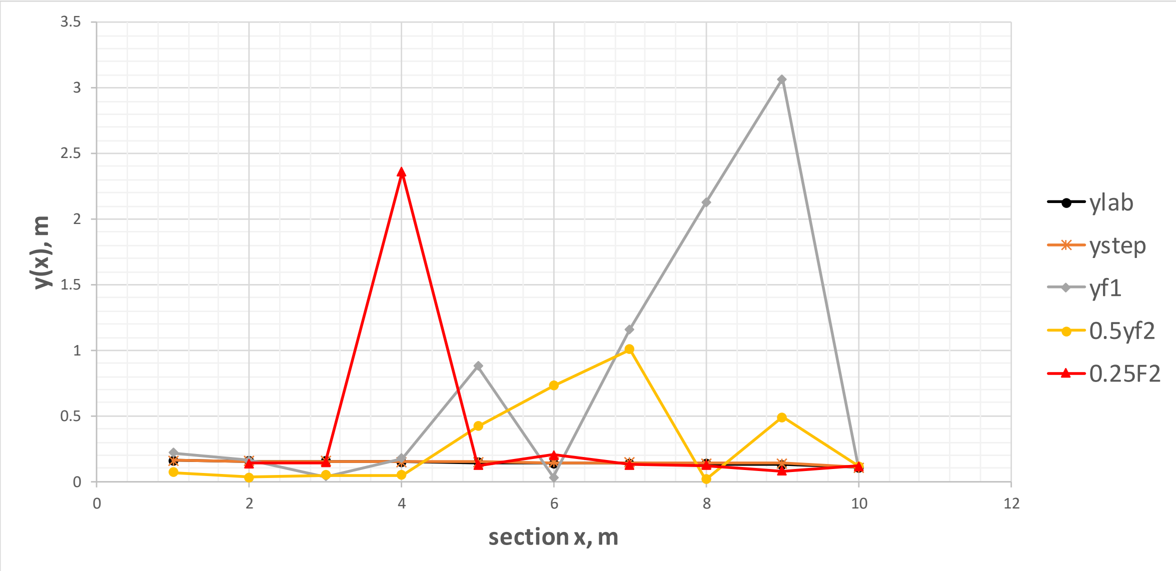

Fig. 2. Comparison between measured and calculated depths. Rectangular channel

From Table 2 and Figure 2 it is observed that the assumption of a behavior of the variation of the depth considering the derivative a mean and under the assumption of a polynomial behavior of degree 2 (factor f_1) of the depth and with respect to the chain x gives high errors, although smaller compared to the assumption of a third degree polynomial behavior (factor f_2); but on average the behavior of $_{0.5}y\ _{f2}$ gave a behavior with less oscillations than $_{yf1.}$ Due to the above, the $_{025}y_{f2}$ calculation was made, which was observed a little more attached to the behavior of the flow in the sections, except in sections 4 and 1 (this last data was not drawn since the figure was deformed a lot when the vertical scale was expanded). The method of the direct step considering the derivative of order one gave the best answer in this case, with respect to the measured laboratory data. There was only almost coincidence in the depth measured with that calculated in the section x = 2 m, with respect to what the direct passage method gave with what the scheme in fractional finite differences with the factor f_1 gave. Higher oscillations were observed in the case of 0.25 and f_2.

4. Conclusions

Considering hypothesis that M profile flow variation from a rectangular channel with laboratory data, with respect to x position corresponds to a derivative of order ½, was not corroborated in this study using fractional finite difference scheme, although result using lowest error was with the Δx = 0.25 m considered, except in two sections of profile in which scheme gave very high depth values. Oscillations were observed in numerical solution scheme that show new calculations are required with other proposals both order γ fractional derivative and Δx in order to seek a reduction of these oscillations and with it derivative order can best represent flow behavior. Up to this point in this line of research, direct step method and order one consideration derivative report the best numerical response with respect to profile measured at laboratory.

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